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Australian Mathematical Society Lecture Series. 3

Introduction to the Analysis of Metric Spaces

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CONTENTS

PREFACE

This text is designed as a basic introductory course in the analysis of metric and normed linear spaces for undergraduate students. It is aimed at providing the abstract analysis components for the degree course of a student majoring in mathematics or an honours student majoring in science or engineering.

It is assumed that such students will have completed a first course in real analysis or a course in calculus which has been carefully developed with attention given to the real analysis foundations. The text

Calculus by Michael Spivak
Benjamin, 1967

is universally acknowledged as presenting just such a calculus course with an eye on rigour.

It is also assumed that the student will have some background in elementary linear algebra.

Such an analysis course as presented here would be included in the lecture programme for a standard undergraduate course no earlier than in the second year. The second year undergraduate programme in Australian universities certainly includes the calculus of functions of many variables, introductory complex analysis and further linear algebra. So it is assumed that such study will be progressing at least concurrently with this course in analysis.

The author follows the educational style set in

Foundations of Modern Analysis by J. Dieudonné
Academic Press, 1960

and which has been the dominant practice in British universities, of studying the analysis of metric spaces in detail before introducing general topological spaces. After all, the spaces which a mathematician is most likely to face are metric spaces and it is preferable from a teaching point of view to work from the more familiar and concrete to the less familiar and abstract. This approach also enables us to gain an early appreciation of the abstract structural nature of much of real analysis: for example, uniform convergence is easier to grasp when we visualise it as convergence in a special function space. But worthwhile applications are also more immediate and this is important in selling the advantages of abstract analysis.

It is important that students majoring in mathematics gain some familiarity with the axiomatic method in analysis for it provides a logically tight investigation of a basically simple abstract structure which manifests itself in a number of diverse examples. The justification for studying abstract metric and normed linear spaces is that these are the fundamental analysis structures underlying many problems in analysis. To demonstrate this we present in the first chapter a wealth of example spaces. In subsequent chapters, after introducing a concept in the general structure we illustrate it in a variety of example spaces.

The mathematician must be able to detect the significant underlying structure in a problem situation and then be able to apply the properties of the general structure to the solution of the specific problem. One of the most spectacular successes in applying metric space theory is in the study of the contraction mapping principle: in the second chapter where we introduce convergence in metric spaces, we go on to demonstrate quite early the great power of the method in applying Banach's Fixed Point Theorem to a range of quite disparate situations. In Chapter IV we aim to communicate some appreciation of the beauty of the existence approach to a proof of the Weierstrass Approximation Theorem and demonstrate the techniques of generalising the proof to a more general situation in the Stone-Weierstrass Theorem. By this stage the student should be impressed with the considerable achievements made by exploiting the abstract structural nature of a problem.

An important feature of this text is that normed linear spaces are introduced from the beginning as a special subfamily of metric spaces. After all most of the metric spaces we deal with in examples are normed linear spaces or subsets of normed linear spaces and it is worthwhile pointing out this structure at the outset. Moreover there are some simplifications which follow from a recognition of normed linear structure: the triangle inequality is easier to handle, the spheres and balls are all translations of scaled images of the unit sphere at the origin and the norm topology accords more with our intuition. Many topics introduced for metric spaces have particular implications for normed linear spaces: for example, with closure we discuss the closure of linear subspaces, with continuity it is natural to enquire into the continuity of linear mappings, and compactness is applied to characterising finite dimensionality.

The basic technique we rely on is the sequential method: for example, our basic definition of cluster points is by convergent sequences, we also point to the importance of the sequential characterisation of continuity of mappings and we derive the properties and applications of compactness from sequential compactness. Furthermore, any explicit reference to the metric topology is deferred to the end of the book. The author has found from teaching courses in the analysis of metric spaces that the early introduction of the notion of open sets and their properties is often an abstraction which side tracks the main thrust of the course without noticeably facilitating the development of the theory. Where a topological account is important, as in a fuller discussion of compactness, it is quite satisfactory to work with the open ball base for the topology. The concluding chapter concerns the metric topology and reviews the earlier material putting it in a topological setting to fit into a further course on the analysis of general topological spaces.

From the earliest development of analysis for normed linear spaces it is natural to enquire about finite dimensional spaces and especially so because the theory is a generalisation from that more familiar situation. It has always seemed rather unnatural to defer a discussion of finite dimensional spaces in general until compactness arguments have been developed. In this text, on the other hand, as each new concept is introduced we determine the particular form it takes for general finite dimensional normed linear spaces as a significant example situation. Using the local compactness of the real number system, we establish that in finite dimensional normed linear spaces, convergence is equivalent to co-ordinatewise convergence and we use this as the fundamental property from which the finite dimensional analysis properties are derived. It is important for the student to understand early that although the study of normed linear spaces springs from finite dimensional spaces the interest in the finite dimensional situation lies essentially in their linear algebra and their being isomorphic to Euclidean or Unitary spaces.

Some topics are notable absent from this course. Connectedness is not mentioned except that we do use the connectedness property of continuous real functions from real analysis. Otherwise it is better dealt with from a topological point of view and is not in general applicable to the course we are considering. We make no reference to example spaces using the Lebesgue integral because it is the opinion of the author that a first course in abstract analysis should not be delayed until the student has studied Lebesgue integration. When integration theory comes to be studied the student should be in a position to appreciate the normed linear space structures which are there.

This text is an introduction to what is traditionally called functional analysis. But here we make no use of the Axiom of Choice in its most general form. The analysis of normed linear spaces is developed up to the point where, with the Axiom of Choice in the form of Zorn's lemma, the Hahn-Banach Theorem can be established. The theory of normed linear spaces as abstract entities begins from that point. Such a study examines duality and the theory of linear operators.

We have not introduced abstract Hilbert spaces and their axiom system reckoning that their special inner product structure finds its major significance not in its metric space analysis but in an exploration of duality and operator theory, subjects which properly belong to a sequel to this text.

In the preface to *Foundations of Modern Analysis*, J. Dieudonné says that he makes "no appeal whatsoever to 'geometrical intuition' at least in the formal proofs ... which we have emphasised by deliberately abstaining from introducing any diagram in the book.". At the time of his writing Dieudonné had a point to make. Now although formal proofs should not rely on diagrams, 'geometrical intuition' does play a vitally important part in creating mathematical ideas and in setting up the rough frame for a proof even in an abstract setting. Moreover, for abstract spaces it is sensible for the student to develop the habit of asking about the form his problem takes, say, in the Euclidean plane, and that often implies a visualisation in terms of Euclidean geometry. To ask a student to work mathematically without using geometrical intuition is like asking him to untie a knot blindfolded. Especially is this so for an abstract analysis course aimed at middle course undergraduates. So this text has a limited number of diagrams which are intended to be geometrically suggestive of the formal material contained in the text.

At the end of each section is a graded selection of exercises which follow generally the order of presentation of the material in the section. A basic difficulty for students beginning a course in abstract analysis is to detect the abstract structural properties in concrete example spaces so to help the student gain facility in doing this, the first exercises in each set are designed to follow a basic concept through several example spaces. Later exercises in each set are conceptually more sophisticated and are intended to introduce a diversity of areas where the general theory is applied.

Although we expect a mathematics text to have a logical development, scarcely ever is such a text read as a novel from beginning to end and certainly it is never reread that way. So a mathematics text without an adequate index is a continual source of frustration to the reader. The index in this text is intentionally detailed giving references to all the significant places where a particular concept is used.

To begin a mathematics text with a preliminary chapter cataloguing prerequisite knowledge is hardly a way to induce initial excitement in the reader. Besides, some or all of such material will be well known by a great many readers. So in this text such material is presented in an appendix.

Most of the material in this lecture course can be covered in between 25-30 lectures given an adequate tutorial programme. However, the material can be readily tailored for various length and content requirements. The preferred use of sequential methods implies that a shorter course using only these methods would be one which concludes in Section 8 before the alternative characterisations of compactness are introduced. Of course the economies of lecturing would demand that not all example spaces be introduced in class and the applications in Section 5 could be curtailed if it is felt that such material overlaps too much with other courses.

However, any course which treats normed linear spaces even in an introductory way as we do here, should discuss linear mappings. A lecturer who proposes to cut Section 7 out of his course should consider the effect of his doing so on the later material in the text.

There are many texts giving more detailed accounts of different sections of the course presented here. Although this text takes its initial inspiration from the approach of J. Dieudonné's book mentioned above, his presentation is generally difficult for a middle year undergraduate student and it covers more ground with more sophistication.

A very useful book written from the same viewpoint but with the development of further analysis of normed linear spaces in mind is

Elements of Functional Analysis by A.L. Brown and A. Page
van Nostrand Reinhold, 1970.

A text on the analysis of metric spaces which is aimed to bring an abstract approach into earlier undergraduate courses is

Real Analysis: an introduction by A.J. White
Addison-Wesley, 1968.

This text is useful in that it puts real analysis in an abstract setting, but there is no explicit reference to normed linear spaces.

Texts such as

Topology and Modern Analysis by George F. Simmons
McGraw-Hill, 1963 and

Topology and Normed Spaces by G.J.O. Jameson
Chapman-Hall, 1974

are both very readable as introductory texts and might be used as references but they develop from a topological point of view.

A text which is very popular with engineers is

Introductory Functional Analysis with Applications
by E. Kreysig
John Wiley & Sons, 1978.

His text contains much more material than ours. Kreysig's aim has been to reduce the topology prerequisites for the course. This will make his text an accessible reference for our course.

The author has given lectures on a course such as this to second year mathematics major students at the University of Newcastle for a period of 10 years. The lecture notes were produced in duplicated form in 1974 but subsequent experience has modified the initial approach considerably to arrive at the form presented here.

Thanks are due to my colleagues in the department for their conversations over the years which have had their effect on the final result, especially Michael Hayes who has always been a steadying influence directing me to the realities of the teaching situation for the average students.

I am indebted to Anne Feletti and to Jan Garnsey who have so competently created typed order out of my handwritten manuscript. I wish to express my thanks to the Editors and referees for their encouragement and assistance in the preparation of this contribution to the Lecture Series of the Australian Mathematical Society.

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