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Excerpt
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I. METRIC SPACES AND NORMED LINEAR SPACES

After some experience with real and complex analysis it becomes apparent that the development of the theory depends to a great extent simply on the notion of distance between the numbers.

For example, in either number system, when we have a sequence $\{x_n\}$ converging to a limit x we commonly write $|x_n - x| \to 0$ as $n \to \infty$ and say that the distance between x_n and x tends to zero as n tends to infinity. Although the same statement applies in the two different number systems, the meaning of the modulus sign $|\cdot|$ depends on the number system where it is used.

This observation suggests that we generalise the analysis of the real or complex numbers to the analysis of metric spaces. A metric space is a non-empty set where the distance between any two points is specified. The notion of distance has to retain those properties of distance used in the real and complex number systems which are evidently vital for the development of a sensible analysis.

1. DEFINITIONS AND EXAMPLES

1.1 Definition. Given a non-empty set X, a distance function d on X, called a *metric* for X, is a function which assigns to each pair of points a real number, (or formally, $d : X \times X \to \mathbb{R}$), satisfying the following properties:
For all $x,y \in X$

(i) $d(x,y) \geq 0$
(ii) $d(x,y) = 0$ if and only if $x = y$
(iii) $d(x,y) = d(y,x)$,

and for all $x,y,z \in X$
(iv) $d(x,y) \leq d(x,z) + d(y,z)$, (the triangle inequality)

A non-empty set X with a metric d is denoted by (X,d) and is called a *metric space*. Different metrics could be defined on the same set giving rise to different metric spaces.

With such a general definition we can expect to have some situations where our metric intuition is strained. The following example is important as such an extreme case. As we introduce new concepts this space will be useful in testing our definitions against our intuition.

1.2 Example. For any non-empty set X the *discrete metric* d is defined by

$$d(x,y) = \left.\begin{array}{ll} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{array}\right\}$$

We may consider this to be the roughest of metrics; given any $x \in X$, it is simply a measure of coincidence with x. □

Most of the metric spaces we will consider are also linear spaces and it is frequently of advantage to take this into account. In most of these cases, the metric is generated by a simpler function called a norm which assigns a length to each vector in the linear space.

1.3 Definition. Given a linear space X over $\mathbb{R}$ (or $\mathbb{C}$), a *norm* $\|\cdot\|$ for X is a function on X which assigns to each element a real number, (or formally, $\|\cdot\| : X \to \mathbb{R}$), satisfying the following properties:
For all $x \in X$

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|\lambda x\| = |\lambda|\,\|x\|$ for any scalar λ,

and for all $x,y \in X$

(iv) $\|x+y\| \leq \|x\| + \|y\|$ (the triangle inequality)

A linear space X with a norm $\|\cdot\|$ is denoted by $(X,\|\cdot\|)$ and is called a *normed linear space*. Again, different norms could be defined on the same linear space giving rise to different normed linear spaces.

1.4 Remark. Given a normed linear space $(X, \|\cdot\|)$, it is clear that the function $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \|x-y\|$$

is a metric for X, and we call this *the metric generated by the norm* $\|\cdot\|$. So then every normed linear space is a metric space under the metric generated by its norm. □

1.5 Examples. $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$

The set of real numbers $\mathbb{R}$ (the set of complex numbers $\mathbb{C}$) is a normed linear space with norm given by the modulus; that is,

$$\|x\| = |x|.$$

We call this the *usual norm* for $\mathbb{R}$ (or $\mathbb{C}$) and it generates the *usual metric*

$$d(x,y) = |x-y|.$$

These are the spaces we are familiar with in real and complex analysis. □

1.6 Examples. $(\mathbb{R}^n, \|\cdot\|_2)$ and $(\mathbb{C}^n, \|\cdot\|_2)$

The set of ordered n-tuples of real numbers $\mathbb{R}^n$ (of complex numbers $\mathbb{C}^n$) is a normed linear space with norm $\|\cdot\|_2$ defined as follows: For $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\|x\|_2 = \sqrt{(|\lambda_1|^2 + |\lambda_2|^2 + \ldots + |\lambda_n|^2)}.$$

We call this the *Euclidean norm* for $\mathbb{R}^n$ (the *Unitary norm* for $\mathbb{C}^n$) and it generates the *Euclidean metric* for $\mathbb{R}^n$ (the *Unitary metric* $\mathbb{C}^n$). We call $(\mathbb{R}^n, \|\cdot\|_2)$ *Euclidean n-space* and $(\mathbb{C}^n, \|\cdot\|_2)$ *Unitary n-space*.

The only norm property which provides any difficulty to verify is the triangle inequality. The proof of this can be derived from the Cauchy-Schwarz inequality.

1.7 Lemma. The Cauchy-Schwarz inequality

In the linear space $\mathbb{R}^n$ *(or* $\mathbb{C}^n$*), for any* $x \equiv (\lambda_1, \lambda_2, \dots, \lambda_n)$ *and* $y \equiv (\mu_1, \mu_2, \dots, \mu_n)$,

$$\sum_{k=1}^{n} |\lambda_k \mu_k| \leq \sqrt{\left(\sum_{k=1}^{n} |\lambda_k|^2\right)} \sqrt{\left(\sum_{k=1}^{n} |\mu_k|^2\right)}$$

Proof. For any positive real numbers a and b

$$2ab \leq a^2 + b^2.$$

Therefore, given non-zero x and y, for each $k \in \{1,2,\dots,n\}$, putting

$$a \equiv \frac{|\lambda_k|}{\sqrt{\left(\sum_{k=1}^{n} |\lambda_k|^2\right)}} \quad \text{and} \quad b \equiv \frac{|\mu_k|}{\sqrt{\left(\sum_{k=1}^{n} |\mu_k|^2\right)}}$$

and summing the consequent inequalities we have

$$\frac{\sum_{k=1}^{n} |\lambda_k \mu_k|}{\sqrt{\left(\sum_{k=1}^{n} |\lambda_k|^2\right)}\sqrt{\left(\sum_{k=1}^{n} |\mu_k|^2\right)}} \leq 1. \ \square$$

Proof of the triangle inequality in 1.6.

$$\begin{aligned} \|x+y\|_2^2 &= \left(\sum_{k=1}^{n} |\lambda_k+\mu_k|^2\right) \\ &\leq \sum_{k=1}^{n} |\lambda_k|^2 + 2\sum_{k=1}^{n} |\lambda_k\mu_k| + \sum_{k=1}^{n} |\mu_k|^2 \\ &\leq \sum_{k=1}^{n} |\lambda_k|^2 + 2\sqrt{\left(\sum_{k=1}^{n} |\lambda_k|^2\right)}\sqrt{\left(\sum_{k=1}^{n} |\mu_k|^2\right)} + \sum_{k=1}^{n} |\mu_k|^2 \\ &\qquad \text{by the Cauchy-Schwarz inequality} \\ &= (\|x\|_2 + \|y\|_2)^2. \ \square \end{aligned}$$

The next two sets of examples are formed by taking different norms on the same underlying linear space $\mathbb{R}^n$ (or $\mathbb{C}^n$).

1.8 Examples. $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{C}^n, \|\cdot\|_1)$

$\mathbb{R}^n$ (or $\mathbb{C}^n$) is a normed linear space with norm $\|\cdot\|_1$ defined as follows:

For $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\|x\|_1 = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|.$$

The triangle inequality follows from the triangle inequality in $(\mathbb{R}, |\cdot|)$:

For any $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $y \equiv (\mu_1, \mu_2, \ldots, \mu_n)$,

$$\begin{aligned} \|x+y\|_1 &= \sum_{k=1}^{n} |\lambda_k + \mu_k| \\ &\leq \sum_{k=1}^{n} |\lambda_k| + \sum_{k=1}^{n} |\mu_k| \\ &= \|x\|_1 + \|y\|_1 . \; \square \end{aligned}$$

1.9 Examples. $(\mathbb{R}^n, \|\cdot\|_\infty)$ and $(\mathbb{C}^n, \|\cdot\|_\infty)$

$\mathbb{R}^n$ (or $\mathbb{C}^n$) is a normed linear space with norm $\|\cdot\|_\infty$ defined as follows:

For $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\|x\|_\infty = \max\{|\lambda_k| : k \in \{1,2,\ldots,n\}\}.$$

We call this the *supremum* (or *uniform*) norm for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

We check that the triangle inequality holds:

For any $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $y \equiv (\mu_1, \mu_2, \ldots, \mu_n)$, if $\max\{|\lambda_k + \mu_k| : k \in \{1,2,\ldots,n\}\} = |\lambda_j + \mu_j|$ then

$$\begin{aligned} \|x+y\|_\infty &= |\lambda_j + \mu_j| \leq |\lambda_j| + |\mu_j| \\ &\leq \max\{|\lambda_k| : k \in \{1,2,\ldots,n\}\} + \max\{|\mu_k| : k \in \{1,2,\ldots,n\}\} \\ &= \|x\|_\infty + \|y\|_\infty . \; \square \end{aligned}$$

We now consider a particular abstract normed linear space where many concrete spaces can be derived as special cases.

1.10 Example. $(B(X), \|\cdot\|_\infty)$

For any non-empty set X we denote by $B(X)$ the set of bounded real (complex) functions on X. $B(X)$ is a real (complex) linear space under pointwise definition of addition and multiplication by a scalar; that is, for any $f,g \in B(X)$,

$$(f+g)(x) \equiv f(x) + g(x) \quad \text{for all } x \in X$$

and for any $f \in B(X)$ and scalar λ

$$(\lambda f)(x) \equiv \lambda f(x) \quad \text{for all } x \in X.$$

$B(X)$ is a normed linear space with norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

We call this the *supremum* (or *uniform*) norm for $B(X)$. We notice that Example 1.9 is the special case when $X \equiv \{1,2,...,n\}$. We check that the triangle inequality holds in the general case:
For any $f,g \in B(X)$,

$$|(f+g)(x)| \leq |f(x)| + |g(x)| \quad \text{for every } x \in X$$

so $$\sup\{|(f+g)(x)| : x \in X\}$$
$$\leq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}. \square$$

Of particular interest are the following two special cases.

1.11 Example. $(m, \|\cdot\|_\infty)$

When $X = \mathbb{N}$, $B(\mathbb{N})$ is the linear space of bounded sequences usually denoted by m (or sometimes ℓ_∞). In this case the norm $\|\cdot\|_\infty$ takes the following form:
For $x \equiv \{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$,

$$\|x\|_\infty = \sup\{|\lambda_n| : n \in \mathbb{N}\}. \square$$

1.12 Example. $(\mathcal{B}(J), \|\cdot\|_\infty)$ and $(\mathcal{B}[a,b], \|\cdot\|_\infty)$

When $X = J$ an interval of real numbers, $\mathcal{B}(J)$ is the linear space of bounded functions on J. We will be specially interested in the particular case when $J = [a,b]$ a bounded closed interval. □

Many significant metric and normed linear spaces are related to others as subspaces.

1.13 Definitions. Given a metric space (X,d) and Y a non-empty subset of X it is clear that d is also a metric for Y, (that is, the restriction of d to $Y \times Y$ is a metric for Y). We denote this restriction by $d|_Y$ and call it the *relative metric* induced by d on Y. We call $(Y, d|_Y)$ a *metric subspace* of (X,d).
Given a normed linear space $(X, \|\cdot\|)$ and a linear subspace Y of X it is clear that the restriction of the norm $\|\cdot\|$ to Y is also a norm for Y. We denote this restriction by $\|\cdot\|_Y$ and call $(Y, \|\cdot\|_Y)$ a *normed linear subspace* of $(X, \|\cdot\|)$.

Very often a metric space where there is no linear structure, can be considered as a metric subspace of a normed linear space. For example, it is quite natural to consider the interval $[a,b]$ as a metric subspace of $(\mathbb{R}, |\cdot|)$ where the usual metric is restricted to $[a,b]$.

The following examples are significant subspaces of those given in Examples 1.11 and 1.12.

1.14 Example. $(c_0, \|\cdot\|_\infty)$

The set c_0 of sequences which converge to zero is a linear subspace of m, and $(c_0, \|\cdot\|_\infty)$ is a normed linear subspace of $(m, \|\cdot\|_\infty)$.

1.15 Examples. $(C(J), \|\cdot\|_\infty)$ and $(C[a,b], \|\cdot\|_\infty)$

The set $C(J)$ of bounded continuous real functions on an interval J is a linear subspace of $\mathcal{B}(J)$. This follows from the algebra of continuous functions in real analysis. In particular, when $J \equiv [a,b]$ a bounded closed interval, since a continuous function on a bounded closed interval is bounded, so $C[a,b]$ is the linear space of continuous real functions on $[a,b]$. Also, since every continuous function on a bounded closed interval has a maximum, the norm $\|\cdot\|_\infty$ on $C[a,b]$ has the form

$$\|f\|_\infty = \max\{|f(t)| : t \in [a,b]\}. \quad □$$

The following example is important in relation to Example 1.15.

1.16 Example. $(C[a,b],\|\cdot\|_1)$

$C[a,b]$ is a linear space with norm $\|\cdot\|_1$ defined by

$$\|f\|_1 = \int_a^b |f(t)|dt.$$

We call this the *integral norm* for $C[a,b]$. The only norm property which provides any difficulty to verify is that part of property (ii) which states that

$$\|f\|_1 = 0 \text{ implies that } f = 0.$$

This follows contrapositively from the following property of continuous functions:
If f is a positive continuous function on $[a,b]$ where there exists a $c \in [a,b]$ such that $f(c) > 0$ then $\int_a^b f(t)dt > 0$. □

1.17 Example. ℓ_2-space

An important generalisation of Euclidean n-space (or Unitary n-space) which retains much of the special structure of such spaces is real (or complex) *Hilbert sequence space* denoted by ℓ_2. The set ℓ_2 whose elements are sequences of scalars $x \equiv \{\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots\}$ such that $\Sigma|\lambda_n|^2$ is convergent, is a linear space under pointwise definition of the linear space operations and is a normed linear space with norm $\|\cdot\|_2$ defined by

$$\|x\|_2 = \sqrt{\left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)}.$$

We can verify simultaneously that ℓ_2 is a linear space and that $\|\cdot\|_2$ is a norm for ℓ_2. The only norm property which provides difficulty to verify is the triangle inequality.

Proof of the triangle inequality. Given any $x \equiv \{\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots\}$ and $y \equiv \{\mu_1,\mu_2,\ldots,\mu_n,\ldots\}$ in ℓ_2 we have from the triangle inequality for Euclidean n-space (Unitary n-space) that, for any $n \in \mathbb{N}$,

$$\sqrt{\left(\sum_{k=1}^{n} |\lambda_k+\mu_k|^2\right)} \leq \sqrt{\left(\sum_{k=1}^{n} |\lambda_k|^2\right)} + \sqrt{\left(\sum_{k=1}^{n} |\mu_k|^2\right)}$$
$$\leq \sqrt{\left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)} + \sqrt{\left(\sum_{k=1}^{\infty} |\mu_k|^2\right)}$$
$$< \infty$$

So $\Sigma|\lambda_n+\mu_n|^2$ is convergent. Consequently $x + y \in \ell_2$ and also

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2. \quad \square$$

<u>1.18 Example</u>. ℓ_p^n-space, $(1 \leq p \leq \infty)$

In Examples 1.6, 1.8 and 1.9 we defined norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on the linear space $\mathbb{R}^n$ (or $\mathbb{C}^n$). These can be considered particular cases of a general class of norms on $\mathbb{R}^n$ (or $\mathbb{C}^n$). For any given $1 \leq p < \infty$ we define the norm $\|\cdot\|_p$ as follows:

For $x \equiv (\lambda_1,\lambda_2,\ldots,\lambda_n)$,

$$\|x\|_p = \left(\sum_{k=1}^{n} |\lambda_k|^p\right)^{\frac{1}{p}}$$

We call this a *p-norm* and refer to $\mathbb{R}^n$ (or $\mathbb{C}^n$) with this norm as real (or complex) *ℓ_p^n-space*.

The only norm property which provides any difficulty to verify is again the triangle inequality. The proof of this inequality, when $1 < p < \infty$, is derived from a generalisation of the Cauchy-Schwarz inequality.

<u>1.19 Lemma</u>. <u>The Hölder inequality</u>.

In the linear space $\mathbb{R}^n$ *(or* $\mathbb{C}$*), given* $1 < p < \infty$*, for any* $x \equiv (\lambda_1,\lambda_2,\ldots,\lambda_n)$ *and any* $y \equiv (\mu_1,\mu_2,\ldots,\mu_n)$,

$$\sum_{k=1}^{n} |\lambda_k\mu_k| \leq \left(\sum_{k=1}^{n} |\lambda_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} |\mu_k|^q\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is derived as for the Cauchy-Schwarz inequality but from the following more general elementary inequality:
For any positive real numbers a and b

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for any given } 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1;$$

(see A.J. White *Real Analysis: an introduction*, p.40). □

Proof of the triangle inequality in 1.18.

For non-zero $x + y$,

$$\sum_{k=1}^{n} |\lambda_k+\mu_k|^p \le \sum_{k=1}^{n} |\lambda_k+\mu_k|^{p-1}|\lambda_k| + \sum_{k=1}^{n} |\lambda_k+\mu_k|^{p-1}|\mu_k|$$

$$\le \Big(\sum_{k=1}^{n} |\lambda_k+\mu_k|^{(p-1)q}\Big)^{\frac{1}{q}}\Big(\sum_{k=1}^{n} |\lambda_k|^p\Big)^{\frac{1}{p}}$$

$$+ \Big(\sum_{k=1}^{n} |\lambda_k+\mu_k|^{(p-1)q}\Big)^{\frac{1}{q}}\Big(\sum_{k=1}^{n} |\mu_k|^p\Big)^{\frac{1}{p}}$$

by the Hölder inequality.

But $(p-1)q = p$ and dividing by $\left(\sum_{k=1}^{n} |\lambda_k+\mu_k|^p\right)^{\frac{1}{q}}$ we have the triangle inequality. □

At this point we should show that Example 1.9 can be included in this class as the case $p = \infty$; this also provides a reason for the notation $\|\cdot\|_\infty$ for the supremum norm.

1.20 Lemma. *In the linear space* $\mathbb{R}^n$ *(or* $\mathbb{C}^n$*), for any given* $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\max\{|\lambda_k| : k \in \{1,2,\ldots,n\}\} = \lim_{p\to\infty}\Big(\sum_{k=1}^{n} |\lambda_k|^p\Big)^{\frac{1}{p}}$$

Proof. For all positive real numbers a and b where $a \ge b$ and $p \ge 1$,

$$a \le (a^p+b^p)^{\frac{1}{p}} = a\Big(1 + \big(\tfrac{b}{a}\big)^p\Big)^{\frac{1}{p}}$$

$$\le a.2^{\frac{1}{p}} \to a \quad \text{as} \quad p \to \infty.$$