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Norbert Wiener
Frontmatter
More information

THE FOURIER INTEGRAL
AND CERTAIN OF ITS APPLICATIONS

THE FOURIER INTEGRAL AND CERTAIN OF ITS APPLICATIONS

BY

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DEDICATED
TO THE MEMORY OF

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Late Professor of Mathematics at the Massachusetts
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FOREWORD

This is a most inspired and inspiring book.

When it was written two of Wiener's major papers had just appeared: Generalized Harmonic Analysis, and Tauberian Theorems. His previous works on potential theory, Brownian motion, Fourier analysis were highly appreciated by a few dozens of mathematicians in the world. It was a happy time in the life of Norbert Wiener. He was thirty-seven years old, he had been married for six years, he had just been promoted to a professorship at the Massachusetts Institute of Technology, and he had spent a pleasant year in Cambridge, England, with his wife Margaret and their small daughters, Barbara and Peggy.

Cambridge (I mean, Cambridge, England) plays a special role in Wiener's life. The beginning of his mathematical career coincided with his meeting with Bertrand Russell in Cambridge in 1913. He had just graduated from Harvard in mathematical logic – at only eighteen – and was supposed to go on in logic with Russell. Actually Russell's advice was to learn more mathematics and physics. Accordingly, Norbert Wiener took courses with Hardy, Littlewood, Mercer, and read Einstein's papers of 1905, and Niels Bohr's recent works. Without any doubt Einstein's theory of Brownian motion had a decisive role in Wiener's inspiration.

The academic year 1931–32 in Cambridge was an opportunity for Wiener to meet Hardy and Littlewood again, to lecture on Fourier integrals (in Hardy's chair) and to discover an ideal collaborator, the young R. E. A. C. Paley. Paley had already worked with Littlewood and with Zygmund, and was to die, one year later, in a skiing accident when visiting Wiener at MIT. Two books resulted from the stay in Cambridge and the collaboration with Paley: the present book, which deals with real variable aspects of the Fourier transform, and its superb companion, *Fourier Transforms in the Complex Domain*, written by Wiener after Paley's death, with both as coauthors.

These two books contain the essential ideas of Wiener in harmonic analysis and should be read by anyone working in the field. The second one also has two chapters on random functions with the most important results of Wiener on Brownian motion. Wiener was still to work and discover new things in harmonic analysis (in particular, on Fourier–Stieltjes transforms). However, his main interests shifted gradually to prediction theory, a topic at the frontier of harmonic analysis, and to cybernetics. Out of the circle of mathematicians Norbert Wiener is known as the founder of cybernetics. However, for mathematicians, his most prominent work is in and around harmonic analysis.

Harmonic analysis as we see it now is essentially harmonic analysis as it was viewed by Wiener. Translations of functions (the term of translation, applied to a function, was introduced by Wiener) and convolutions (not yet named in English, for Wiener uses the German word *Faltung*) play a basic role. From the point of view of physicists and engineers, convolutions appear whenever a signal is transformed into an observation through an apparatus: the convolution kernel defines the action of the apparatus. From a mathematician's viewpoint regularization of functions and summation processes for trigonometric series or integrals are nothing but convolutions.

Convolutions appear also, in a hidden form, when some summation processes are applied to ordinary series, integrals, or mean values. Wiener had examples coming from his theory of generalized harmonic analysis. His friend A. E. Ingham (another connection with Cambridge!) led him to a beautiful theorem of Littlewood (generalizing a rather simple result of Tauber) which provides another example. Finally, the prime number theorem can be approached in the same manner. The unifying tool is what Wiener called, as a homage to Littlewood, the general Tauberian theorem.

The Tauberian theorem states that when a limit exists in a certain way, it exists also in another way. What defines the way is a convolution kernel. In a more abstract form, the Tauberian theorem says that, under some conditions, the second kernel belongs to the span of the translates of the first (what Wiener calls its 'extension'). Now comes the Fourier transform. If the Fourier transform of the first kernel does not vanish, its extension contains all kernels. In other words, $L^1(\mathbb{R})$ is generated by the translates of a function whenever its Fourier transform does not vanish. To prove this, a detour through Fourier series (looking at $L^1(\mathbb{Z})$ instead of $L^1(\mathbb{R})$) proves convenient: the Fourier transforms become sums of absolutely convergent Fourier series (the class A of Wiener) and the basic lemma says that, whenever a function belongs to A and does not vanish, its inverse belongs to A.

Not only did the general Tauberian theorem give a unifying view on questions involving summations and limits, but it introduced a paradigm for what was called abstract harmonic analysis a few years later. The class A is a normed ring in the sense of Gelfand, a Banach algebra as we say now. Analytic functions operate on A (this is the Weiner–Lévy theorem) and questions about endomorphisms of A, operating functions, structure of closed ideals, became famous in the years 1940–60. Wiener converted absolutely convergent Fourier series into a central subject in analysis.

The Tauberian theorems occupy a central place in the present book. The last application contains the first motivation, a formula suggested by generalized harmonic analysis. Generalized harmonic analysis is the subject-matter of the last chapter, though it was conceived before the Tauberian theorems. Roughly speaking, it consists in studying a large class of functions for which a 'spectrum' can be defined. Already Bohr's and Besicovitch's theories of almost periodic functions had given a good mathematical model for natural phenomena with discrete spectra. However, Wiener argues that most spectra occurring in physics are continuous, or contain both a discrete and a continuous part. Wiener's model consists of functions $f(x)$ for which the mean value of $f(x)\,\overline{f(x+\xi)}$ exists whatever the translation ξ may be. This mean value, $\phi(\xi)$, divided by $\phi(0)$, expressed a kind of correlation between the function and its translate by ξ. Bochner's theory of positive definite functions applies to ϕ and proves that ϕ, when it is continuous, is the Fourier transform of a positive measure, exactly the spectrum of Wiener. Actually Bochner's theory was not available to Wiener, and he had to build up his own method in order to exhibit the spectrum. The example he gives of a function with a continuous spectrum is a random function – one of the first examples of probability methods applied to a problem in analysis. The end of the chapter deals with Bohr's almost periodic functions, proving in particular the Plancherel formula for these functions. Plancherel's formula expresses an L^2-isometry which is like Ariadne's thread, running through Wiener's mathematical world.

Plancherel's formula in the context of $L^2(\mathbb{R})$ is the subject of the first chapter of the book. The original treatment of Wiener through Hermite functions was later the basis for the theory of homogeneous chaos: here again we see the connection between his views on stochastic processes and questions in harmonic analysis. Actually the so-called Wiener integral, like the Fourier transform, has the fundamental property of defining an isometry in L^2-spaces. A modern introduction of Brownian motion can start from this fact.

When Wiener's book appeared, Fourier series and integrals were a rather popular subject in mathematics. Several books by first class mathematicians contributed to the field. Wiener's book has a special flavour. It is not really a treatise on the Fourier integral. It is not intended to pack and dispatch as much material as possible. It does not look for the shortest ways to prove theorems. It is highly original, with emphasis on a selection of important topics, with unifying views and methods, large perspective and careful attention to details. It is a most attractive expression of his personality.

This is the third printing of the book. Wiener would have welcomed it. He was very sensitive to the reactions of mathematicians towards his papers and he resented it whenever there was a lack of recognition of his work. It might have been the case with the reviews on this book. Surprisingly, it got only eighteen lines of analysis and comment in *Zentralblatt für Mathematik* when it appeared, and two lines in *Mathematical*

Reviews for the second printing, while very poor papers are reviewed at great length. No matter what kind of reviews it got, this little book is one of the great mathematical achievements of the century.

November, 1987 Jean-Pierre Kahane

PREFACE

The present book is in substance an elaboration of a course of fifteen lectures on the Fourier Integral and its Applications, given at the University of Cambridge during the Lent Term of 1932. When I arrived in Cambridge during the Michaelmas Term of 1931, on leave of absence from the Massachusetts Institute of Technology, I had vague plans of writing up certain topics in the theory of harmonic analysis into a book on the subject. My original idea was of a rather comprehensive treatise, proceeding from the elements of Lebesgue integration through the L_2 theory of the Fourier series to the Plancherel theorem, the Fourier Integral, the periodogram, and lastly, to theorems of Tauberian type. My impulse to write a book of this type arose from a dissatisfaction with the preponderant rôle of convergence theory in existing textbooks on the subject, and from the need for a treatment more in line with the extensive periodical literature.

As far as my desire to write a book sprang from the need for a textbook to use in my course at the Massachusetts Institute of Technology, it has largely been dissipated by the recent appearance of a book on the Theory of Functions by Professor Titchmarsh. Several chapters of his book are devoted to the treatment of Fourier series from the modern point of view. Unfortunately—from my standpoint—he does not allot a great deal of space to the Fourier Integral and related matters. Thus, while there is now no need for the comprehensive treatise which I at first contemplated, there is need for a discussion of the Fourier Integral from the modern point of view. When Professor Titchmarsh's book has been in use for some five years, and has become the basis for higher instruction in Fourier series, it will be possible to treat the Fourier Integral in a thoroughgoing and coordinate way, but for the present we shall have to content ourselves with more fragmentary treatments.

Thus when Mr Besicovitch and Professor Hardy suggested my giving a course during the Lent Term, I gladly fell in with their plans, and offered as a topic the Fourier Integral and its Applications. It was none of my purpose to aim at completeness, but merely to present various aspects of the theory whose sole unity was that I had worked in all of them. When Professor Hardy later suggested that I should submit the manuscript to the Cambridge University Press, we both agreed that at this time it was better to make the book a frank course of lectures than to strive for the clean-cut outline of a treatise. There are three more or less separate groups of ideas which accordingly find representation: the group pertaining to the Fourier transform and the Plancherel theorem; the notions of an absolutely convergent Fourier series and of a Tauberian theorem; and the concept of the spectrum. This last idea is, it is true, dependent on both the preceding parts of the book, and serves to give it some degree of unity.

The main results of the book may be listed as follows:

(1) The Plancherel theory of the existence of the Fourier transform of a function of L_2, together with the associated Parseval theorem and the proof of the theorem yielding the inverse of a Fourier transformation;

(2) The theorem asserting that if $f(x)$ is a continuous non-vanishing function with an absolutely convergent Fourier series, the Fourier series of $1/f(x)$ converges absolutely;

(3) Various forms of general Tauberian theorems;

(4) The Lambert-Tauber theorem and the de la Vallée Poussin-Hadamard theorem concerning the distribution of primes;

(5) The Ikehara-Landau theorem and its application to the distribution of primes;

(6) The theorem connecting the mean of the square of the modulus of a function with a singular quadratic form in its integrated Fourier transform;

(7) The theorem that a function which has a spectrum has a *positive* spectrum;

(8) A group of theorems concerning the spectra of linear transforms of a given function;

(9) The Weierstrass and Parseval theorems for almost periodic functions.

Naturally, most of these topics have already been treated in various monographs by myself and others. These are given in a bibliography at the end, and my own monographs will not necessarily be cited elsewhere.

After some deliberation, I decided to omit a discussion of harmonic analysis in the complex plane, on the ground that it required too much introductory material not needed elsewhere in the book, and would throw it out of balance.

The papers leading up to this book have been read in proof and most helpfully criticized by Professor J. D. Tamarkin of Brown University and by several of my colleagues and students. To these I wish to express my thanks. I wish to thank my Cambridge colleagues and the Syndics of the Cambridge University Press for causing this book to be written and making its publication possible. Particular gratitude is due to Professor Hardy of Cambridge and to Mr Skewes of the University of the Cape of Good Hope for their painstaking reading of the manuscript.

N. W.

Cambridge
July 1932