INTRODUCTION

§ 1. The Nature of Harmonic Analysis.

In the hierarchy of branches of mathematics, certain points are recognizable where there is a definite transition from one level of abstraction to a higher level. The first level of mathematical abstraction leads us to the concept of the individual numbers, as indicated for example by the Arabic numerals, without as yet any undetermined symbol representing some unspecified number. This is the stage of elementary arithmetic; in algebra we use undetermined literal symbols, but consider only individual specified combinations of these symbols. The next stage is that of analysis, and its fundamental notion is that of the arbitrary dependence of one number on another or of several others—the function. Still more sophisticated is that branch of mathematics in which the elementary concept is that of the transformation of one function into another, or, as it is also known, the operator. It is only in connection with the operational calculus that the true significance of harmonic analysis is to be appreciated.

Let us consider, then, an operation T, transforming functions $f(x)$ defined over $(-\infty, \infty)$ into functions $T\{f\} = g(x)$, defined over the same range. If

$$T\{f_1(x) + f_2(x)\} = T\{f_1(x)\} + T\{f_2(x)\};$$
$$T\{af(x)\} = aT\{f(x)\} \qquad (a \text{ constant});$$

the operator $T\{f\}$ is said to be *linear*. Linear operations are common in physics—commoner indeed in physics than in nature, for the first approximation to a most definitely non-linear operator is often a linear one. An even more common type of operator has been given by Volterra the somewhat inappropriate name of an *operator of the closed cycle*. It has the property that if $T\{f\} = g$, and if $f(x+h) = f_1(x)$, $g(x+h) = g_1(x)$, then

$$T\{f_1(x)\} = g_1(x).$$

If the argument x is taken to be the time, then most of the operators of physics are of the closed cycle, for there are extremely few physical processes in which a change in the time of commencement has any further effect than a corresponding change in the time at which any given stage of the process takes place.

The function e^{iux} plays a singularly important rôle with respect to operators of the closed cycle. This results from the fact that

$$e^{iu(x+h)} = e^{iuh} e^{iux},$$

or in words, that a time displacement of such a function produces no other change in it than to multiply it by a complex number of modulus 1. If the operator T which we are considering is both linear and of the closed cycle, it becomes a matter of interest to consider the result of applying it to linear combinations of functions such as e^{iux}. Let us note that

$$T(e^{iu(x+h)}) = T(e^{iuh} e^{iux}) = e^{iuh} T(e^{iux}),$$

so that if we put $T(e^{iux}) = \phi(x)$, we have

$$\phi(x+h) = e^{iuh} \phi(x),$$

or

$$\phi(h) = \phi(0) e^{iuh}.$$

Thus the effect of T on e^{iux} is merely to multiply it by a constant. It follows that

$$(1{\cdot}1) \qquad T\left(\sum_1^N a_n e^{iu_n x}\right) = \sum_1^N a_n b_n e^{iu_n x},$$

where the b_n's depend only on u_n and T, not on a_n. In other words, if we regard the set of coefficients a_n as in some way representing the function $\sum_1^N a_n e^{iu_n x} = f(x)$, the linear operation T of the closed cycle applied to $f(x)$ corresponds to the multiplicative factors b_n applied to a_n. This fact and others that are its formal generalizations constitute the chief significance of methods of harmonic analysis.

In Chapter I our chief purpose is to extend theorems of the type (1·1) to the case where $\sum_1^N$ is replaced by an integral of the appropriate kind. We investigate some of the conditions under which a function $f(x)$ determines a function $g(u)$ given formally by

$$(1{\cdot}2) \qquad g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx,$$

and find that these conditions yield a result which we may formally write

$$(1{\cdot}3) \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, e^{iux}\, dx,$$

which may be regarded as an extended form of $f(x) = \sum_{1}^{N} a_n e^{iu_n x}$. If we put

$$(1{\cdot}4) \qquad g_1(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\, e^{-iux}\, dx,$$

we shall obtain

$$(1{\cdot}5) \qquad \int_{-\infty}^{\infty} f_1(x-\xi) f(\xi)\, d\xi = \int_{-\infty}^{\infty} g_1(u)\, g(u)\, e^{iux}\, du.$$

The operator which turns $f(x)$ into $\int_{-\infty}^{\infty} f_1(x-\xi) f(\xi)\, d\xi$ thus corresponds formally to the operator of multiplication by $\sqrt{2\pi}\, g_1(u)$ applied to g.

In Chapter II we devote our attention to the asymptotic behaviour of such linear closed-cycle transforms of $f(x)$ as

$$(1{\cdot}6) \qquad \int_{-\infty}^{\infty} K(x-\xi) f(\xi)\, d\xi.$$

This forms the subject-matter of the Tauberian theory—often in a form disguised by a change of variable, in which (1·6) is replaced by

$$\frac{1}{x} \int_{0}^{\infty} Q(x/\xi)\, \phi(\xi)\, d\xi.$$

It is not surprising, in view of the likeness of (1·6) to the left-hand expression in (1·5), that we shall find the function

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x)\, e^{-iux}\, dx$$

playing an important rôle in this theory. This is all the more natural, in consideration of the fact that the asymptotic properties of a function are independent of any choice of origin, and are hence invariant under any transformation $x_1 = x + h$, which also leaves invariant all operators of the closed cycle.

In Chapter III we return to the consideration of expressions of a similar type to (1·3), but now adapted to the treatment of functions $f(x)$ which need not be small in some sense at infinity, as are those of Chapter I. The theory developed in Chapter II appears as a useful tool. One application of great importance is to the Bohr theory of almost periodic functions. We shall term the transformation which yields $f(x+\lambda)$ when applied to $f(x)$ (λ real) a *translation*. The conception of almost periodicity, like that of periodicity, involves no reference to an origin, and therefore has that type of invariance under translation which makes relevant the consideration of operators of the closed cycle and of an harmonic analysis in terms of the function e^{iux}.

§ 2. The Properties of the Lebesgue Integral.

It is well known that an adequate theory of the Fourier series can only be established on the basis of Lebesgue integration. All those theorems which proceed from a given function to its Fourier coefficients can indeed be established on the basis of any less inclusive concept, such as that of the Riemann integral, but the fundamental theorem which proceeds from a set of coefficients to the existence of a function having these Fourier coefficients, that of Riesz and Fischer, is simply false for any definition narrower than that of Lebesgue. In the theory of Fourier series, the function to be expanded and its coefficients are two very different sorts of things—the function is defined for a continuous infinity of values, but is periodic, and hence need only be given within a single period, whereas the coefficients are a discrete non-periodic set of numbers. In the case of the Fourier integral, however, if the function to be expanded is $f(x)$ of (1·3), the function $g(u)$ plays the part of the set of Fourier coefficients, and the formal similarity between (1·2) and (1·3) shows how impossible it is to say which of f and g is the function we are expanding and which is the set of coefficients. Both are non-periodic functions defined over continuous infinite ranges. Accordingly, it is impossible to segregate the difficulties of the theory of the Fourier integral into two categories, but instead we find that all the difficulties which in the Fourier series arise either on proceeding from the function to the coefficients or from the coefficients to the function are here met with in both arguments. It is totally impossible to establish a reasonably complete and symmetrical theory of the Fourier integral except on the basis of Lebesgue integration.

It is no part of the present book to attempt a thorough and systematic treatment of the Lebesgue integral. However, in view of the none too well-established position of the theory in the curriculum of English and American universities, it has seemed advisable to give a résumé of the definitions of Lebesgue measure and of the Lebesgue integral, together with a few of the cardinal theorems which come into constant use. In the choice of these theorems, I have been greatly influenced by the choice made for similar purposes by Professor Hardy in his lectures on Fourier series. It is my intention to make the selection full enough for this book to be intelligible to anyone who is fairly grounded in the elementary non-Lebesgue theory of the functions of a real variable. Naturally, none but the extremely indolent will content himself with taking the proof of these fundamental theorems on faith.

Let us start then with a set of points S on a finite interval (a, b). Let us inclose all points of S in a finite or denumerable set of intervals of total length M. The length M may of course be given different values by different choices of the set of inclosing intervals. We shall call the greatest lower bound of possible values of M the *outer measure* of S, and shall write it $\overline{m}(S)$.

Now let $\overline{S}$ be the set of all those points of (a, b) that do not belong to S. Let us put

$$\underline{m}(S) = b - a - \overline{m}(\overline{S}).$$

We shall term $\underline{m}(S)$ the *inner measure* of S. In case

$$\underline{m}(S)=\overline{m}(S),$$

we shall write $m(S)$ for the common value of these two quantities, shall term it the *measure* of S, and shall say that S is *measurable*. It is easy to show that the measure and the measurability of S do not depend on the length of the interval (a, b).

On an infinite interval, we shall term a set S measurable if whenever $a<b$, the portion of S in (a, b) is measurable. If we write $S(a, b)$ for this portion of S, we shall put

$$m(S)=\lim_{\substack{a\to-\infty \\ b\to\infty}} m(S(a, b)).$$

In two or more dimensions, the definition of measure is only changed in so far as we replace "interval" by "rectangle" or "rectangular parallelepiped" and "length" by "area" or "volume," in the sense appropriate to the number of dimensions in question. In the definition of the measure of a set in the whole plane or in all space, the interval (a, b) is replaced by an analogue, which is allowed to grow in all directions. The definitions of measure thus obtained are not restricted if we specify the orientation of our rectangles and rectangular parallelepipeds, but are independent of this orientation.

We shall define a *null set* as one of measure 0. A proposition involving a variable point is said to be true *almost everywhere* if it only fails to be true over (at most) a null set.

A function $f(x)$ is said to be *measurable* over (a, b) (a and b finite) if the set of values of x on (a, b) for which $\alpha \leqslant f(x) \leqslant \beta$ is measurable for every α and β. If the function $f(x)$ is k over the measurable set of points S lying within (a, b) and is zero elsewhere, we define the *Lebesgue integral*—we shall simply say *integral*—of $f(x)$ over (a, b) to be

$$\int_a^b f(x)\,dx = km(S). \tag{2·01}$$

If $f(x)=\sum_1^n f_k(x)$, where each one of the functions $f_k(x)$ is of the type for which the integral is defined in (2·01), we shall write

$$\int_a^b f(x)\,dx = \sum_1^n \int_a^b f_k(x)\,dx.$$

For any real bounded function of $f(x)$, we shall define

$$\overline{\int_a^b} f(x)\,dx,$$

the *upper integral* of $f(x)$, as the greatest lower bound of

$$\int_a^b g(x)\,dx,$$

where $g(x)$ is a function of the type for which the integral is defined in (2·01) and $g(x) \geqslant f(x)$ over (a, b). In the same circumstances, the lower integral is defined by

$$\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b}(-f(x))\,dx.$$

In case

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

we shall write $\int_a^b f(x)\,dx$ for their common value.

We now introduce a pair of notations to which we shall have frequent occasion to return in later paragraphs. We put

$$f_{A,B}(x) = A\,(f(x) < A);\; = f(x)\,(A \leqslant f(x) < B);\; = B\,(B < f(x));$$

and

$$f_A(x) = f(x) \quad (|f(x)| < A); \quad = Af(x)/|f(x)| \quad (|f(x)| > A).$$

Let us note that the second definition yields $f_{-A,A}(x)$ if $f(x)$ is real, but is significant even in the case of complex-valued functions. Let us further note for future reference that in all cases

$$(2{\cdot}015) \qquad |f_{A,B}(x) - f_{A,B}(y)| \leqslant |f(x) - f(y)|;$$
$$|f_A(x) - f_A(y)| \leqslant |f(x) - f(y)|.$$

We now may complete the definition of Lebesgue integration for real unbounded functions. We define $\int_a^b f(x)\,dx$ by

$$\int_a^b f(x)\,dx = \lim_{\substack{A \to -\infty \\ B \to \infty}} \int_a^b f_{A,B}(x)\,dx.$$

In case the integral of $f(x)$ as thus defined exists, we shall say that $f(x)$ is integrable, or that it belongs to the class L_1.

Let us now enunciate certain propositions concerning measure and integration to which we shall have subsequent occasion to refer. The proofs are to be found in any standard treatise on the Lebesgue integral. We shall indicate these propositions by the letter X, as well as certain theorems which we shall prove but which belong rather to the background of the theories of this book than to the theories themselves.

X_1. *If S is the logical sum of the finite or denumerable set of non-overlapping measurable sets of points $S_1, S_2, \ldots, S_n, \ldots$, then S is measurable, and*

$$m(S) = m(S_1) + m(S_2) + \ldots + m(S_n) + \ldots.$$

X_2. *The logical sum and the logical product of a finite or denumerable set of measurable sets are measurable.*

X_3. *If $S_1, \ldots, S_n, \ldots$ is a sequence of measurable sets with logical product S, and S_k contains S_{k+1} for all values of k, then*

$$m(S) = \lim_{k \to \infty} m(S_k).$$

X_4. *If S_1 and S_2 are measurable sets, and S_1 contains S_2, then*

$$m(S_1) \geqslant m(S_2).$$

X_5. *The logical sum of a finite or denumerable set of null sets is a null set.*

X_6. *If $f(x)$ is a measurable function, so is $|f(x)|$, and if $f(x)$ and $g(x)$ are measurable functions, so are $f(x)\,g(x)$, and $f(x)/g(x)$, and $\alpha f(x) + \beta g(x)$, where α and β are any real constants, provided these functions are well-defined except on a null set.*

X_7. *The various definitions of Lebesgue integration given for different classes of functions are consistent, in the sense that where two are applicable, they yield the same value.*

X_8. *If $f(x)$ is a measurable function and $g(x)$ is an integrable function, and $|f(x)| \leqslant |g(x)|$ for all x, then $f(x)$ is integrable. In particular, all bounded measurable functions are integrable.*

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X_9. *If α and β are real constants, and $f(x)$ and $g(x)$ are integrable, so is $\alpha f(x)+\beta g(x)$. We have*

$$\int_a^b [\alpha f(x)+\beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

X_{10}. *If $f(x)$ is a non-negative integrable function,*

$$\int_a^b f(x)\,dx \geqslant 0.$$

As corollaries, we have for any integrable f,

$$\int_a^b |f(x)| \geqslant 0;$$

for $f \geqslant g$,

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx;$$

and in general

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx \leqslant |b-a| \limsup_{a\leqslant x\leqslant b} |f(x)|.$$

X_{11}. *If $\{f_n(x)\}$ is a sequence of integrable functions, if*

$$f_n(x) \leqslant f_{n+1}(x)$$

for every x on (a, b) with the possible exception of a null set, and if

$$I = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

is finite, then, with the possible exception of a null set of values of x, the limit

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists and is finite, and

$$I = \int_a^b f(x)\,dx.$$

As a corollary, if $\sum_1^\infty g_n(x)$ is a series of positive integrable functions, it may be integrated term by term, in the sense that if

$$\sum_1^\infty \int_a^b g_n(x)\,dx$$

converges, then $\sum_1^\infty g_n(x)$ converges almost everywhere, and

$$\int_a^b \left[\sum_1^\infty g_n(x)\right] dx = \sum_1^\infty \int_a^b g_n(x)\, dx. \tag{2·02}$$

This is known as the test of *monotone convergence* for the term-wise integrability of a series.

X$_{12}$. *If* $\{f_n(x)\}$ *is a sequence of integrable functions, if there exists an integrable function* $F(x)$ *independent of* n *such that*

$$|f_n(x)| \leqslant F(x)$$

for all values of x *on* (a, b), *and if*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

almost everywhere, then $f(x)$ *is integrable, and*

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

In particular, $F(x)$ may be a constant. Applying this result to series, we see that if $\sum_1^\infty g_n(x)$ is a series of integrable functions with uniformly bounded partial sums, or with partial sums uniformly dominated by an integrable function $F(x)$, and if it converges almost everywhere, then it may be integrated term by term, in the sense that both sides of (2·02) will exist and will have the same value. This is known as the test of *dominated convergence* for the term-by-term integrability of a series, or in the case where $F(x)$ is constant, the test of *bounded convergence*.

X$_{13}$. *If* $f(x)$ *is integrable,* $\int_a^x f(\xi)\, d\xi$ *is a continuous function of its upper limit of integration* x. *We have almost everywhere*

$$f(x) = \frac{d}{dx} \int_a^x f(\xi)\, d\xi.$$

This is known as the fundamental theorem of the calculus. Another form of it asserts that

$$\frac{1}{\epsilon} \int_x^{x+\epsilon} f(\xi)\, d\xi \to f(x)$$

almost everywhere as $\epsilon \to 0$.