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W. V. D. Hodge
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THE THEORY AND APPLICATIONS OF HARMONIC INTEGRALS

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BY

W. V. D. HODGE, M.A., F.R.S.

Lowndean Professor of Astronomy and Geometry,
and Fellow of Pembroke College,
Cambridge

With a foreword by Sir Michael Atiyah, F.R.S.

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FOREWORD

Hodge's book on Harmonic Integrals is one of the great landmarks of twentieth-century mathematics. Both in detail and in general outlook it set the stage for the global approach which has dominated geometry ever since. Hodge's work immediately attracted the attention of the leading figures of the time, including Lefschetz and Weyl who recognized its importance. The intervening decades have only served to reinforce their views, and Hodge theory continues to occupy a central position in contemporary research.

It is the fate of the pioneer to be the victim of his own success. Succeeding generations embellish and polish the first primitive steps so that future students will come to see the work as 'almost obvious'. Such is the back-handed tribute which posterity pays to its great innovators. For this reason the modern reader will find the technical presentation in Hodge's book rather cumbersome and pedestrian, but the classics are there to take us back to the birth of a subject, to show us the genesis of ideas as they appeared at the time. They are not meant to be vehicles of contemporary instruction.

When Hodge started his research in the early thirties he had before him the works of Picard and Lefschetz on algebraic integrals and their periods, together with the great body of classical algebraic geometry of the Italian school. His first breakthrough was his proof, following ideas of Lefschetz, that a (non-zero) double integral of the first kind (i.e. a holomorphic 2-form) on an algebraic surface has non-trivial periods. Ironically Lefschetz at first refused to accept the proof and campaigned fiercely against it. Eventually he was converted and became Hodge's strongest supporter.

Hodge soon realized that the main difference between curves and surfaces was that in the latter case holomorphic forms and their conjugates do not exhaust the cohomology.

The reason, as we now know, is the existence of harmonic forms of type (1,1) and Hodge's ideas gradually evolved in this direction. In due course Hodge proved his index or signature theorem which gave a topological interpretation to the geometric genus p_g (the dimension of the space of holomorphic 2-forms). Hodge's formula can be written as

$$b_2^+ = 2p_g + 1$$

where $b_2 = b_2^+ + b_2^-$ is the decomposition of the second Betti number according to the signature of the intersection matrix of 2-cycles. This signature formula was the first great success of Hodge theory, and it made a big impact.

The striking thing about the Hodge signature theorem is that the result is intrinsic to the geometry and topology of the surface but its proof requires the introduction of an auxiliary tool, namely a Kähler metric. This, in the end, is the justification for the introduction of real analysis into algebraic geometry.

More generally the Hodge numbers $h^{p,q}$ (the dimension of the space $H^{p,q}$ of harmonic forms of type (p,q)) are invariants of algebraic varieties, independent of the choice of Kähler metric. Only with the introduction of sheaf cohomology in the fifties were the $h^{p,q}$ given an intrinsic definition, namely

$$h^{p,q}(X) = \dim H^q(X, \Omega^p)$$

where Ω^p is the sheaf of holomorphic p-forms on X.

With the enormous explosion of algebraic geometry since the introduction of sheaf theory, Hodge theory has continued to play a key role. The relation between the (p,q) decomposition and the integral lattice of homology gives an elaborate structure to the period relations. This has been extensively studied by Griffiths and subsequently by Deligne who has coined the term 'Hodge structure' to describe these period relations.

Although Hodge's motivation came from algebraic geometry and this is the area in which it has had its deepest and most striking applications, the theory does of course apply in the first instance to general Riemannian manifolds. As such it is part of global differential geometry and there have been notable applications in this context. The famous Bochner 'vanishing theorems' provide a direct link between positivity conditions on curvature and vanishing of Betti numbers (the extension by Kodaira to Kähler manifolds is also important in algebraic geometry). Also harmonic forms have proved useful in connection with Lie groups. This was already recognized, for compact groups, by Hodge and figures as the last chapter of his book. Subsequently applications to discrete subgroups of non-compact Lie groups have been made by applying Hodge theory to compact locally symmetric spaces.

Also, in a Riemannian context, Hodge theory has a parallel in the analysis of the Dirac equation and the study of harmonic spinors. These were studied extensively by Atiyah and Singer in connection with the index theory of elliptic differential operators and they are also important in present-day physics.

In physical terms Hodge theory has been reinterpreted by Witten in the framework of super-symmetric quantum mechanics. The Hodge Laplacian is the Hamiltonian of the quantum mechanical system. This use of Hodge theory in physics appears to be quite far-reaching and it extends, at least formally, to the context of certain quantum field theories. According to Witten these should be viewed as Hodge theory on appropriate infinite-dimensional manifolds.

In a different but related direction Hodge theory has had a remarkable generalization in the study of the Yang–Mills equations. These are non-linear analogues of the Hodge equations and they have been exploited with spectacular success by Donaldson for the study of 4-dimensional manifolds. Moreover, Donaldson imitates Hodge by using Kähler metrics on algebraic surfaces to define invariants which are then shown to have a purely algebro-geometric significance. This work of Donaldson can thus be viewed as a non-linear generalization of the Hodge signature theorem. In fact the signature of the algebraic surface (or rather b_2^+) enters into the dimension formula for the Donaldson moduli spaces of solutions of the Yang–Mills equations precisely through their linearization.

This very brief review hardly does justice to the scope of Hodge theory at the present time but it does indicate its richness and variety. The connections with theoretical physics would have particularly pleased Hodge since he had been much influenced by Maxwell's equations and was for many years a colleague of Dirac.

Michael Atiyah
1988

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PREFACE

The subject of this book is the study of certain integrals defined in a type of space which is of importance in various branches of mathematics. The space is locally the space of classical Riemannian geometry, and in the large it is an orientable manifold. By considering simultaneously the local and general properties of the space, we are led to the study of a class of integrals in the space to which the name harmonic integral has been given.

In its original form the book formed a section of an essay for which the Adams Prize of 1936 was awarded, but since then it has been revised and entirely re-written, and the subject-matter has been enlarged by the addition of a chapter dealing with the application of the harmonic integrals to the theory of continuous groups.

The first chapter is concerned with the geometry of the space in which the integrals are defined. The properties which are required for the work of later chapters fall into two classes, those relating to the differential geometry of the space, and those which are topological properties. Both these subjects are treated at length in a number of standard works, and there seems no reason to add to the existing literature. I have therefore contented myself with a brief survey of the bare essentials, but I hope that I have said enough on these topics to enable the reader who is unacquainted with Riemannian geometry or topology to understand the later chapters. The second chapter deals with the properties of integrals on a manifold, and in it I give a proof of de Rham's theorem on the existence of an integral with assigned periods, while the third chapter introduces harmonic integrals and contains a proof of the fundamental existence theorem for these integrals.

The remainder of the book is concerned with the applications of the theory of harmonic integrals to other branches of mathematics. It is clear that applications of our theory will be possible in any field of mathematical research in which a Riemannian manifold plays a part. But when the differential geometry of the manifold has special properties we are able to go much further with our theory than in the general case. I have not attempted to invent any manifolds in which the conditions are particularly favourable for the development of the properties of harmonic integrals, and I have confined my attention to manifolds which arise naturally in two important branches of mathematics.

In Chapter IV I consider the properties of harmonic integrals in the Riemannian of an algebraic variety. It is necessary to introduce a metric which is irrelevant in the classical theory of varieties, but the greater part of the chapter is devoted to deducing, from the properties of harmonic integrals, invariants of the manifold which do not depend on the metric. Most of the results obtained belong to the transcendental theory of varieties, but a few geometrical results can be deduced by the methods which we employ. But, while it is possible to explain the results belonging to the transcendental theory without requiring much knowledge of the theory of algebraic varieties on the part of the reader, a considerable knowledge of algebraic geometry is required in order to understand the significance of the geometrical applications of our theory. Since these applications are at present somewhat isolated, and can only be regarded as preliminary, there does not seem to be sufficient justification for prefacing this part of the chapter with a lengthy account of the algebraic geometry of varieties. I have therefore confined my account of the geometrical applications of harmonic integrals to two paragraphs [§§ 51, 52] which really form an appendix to the chapter, and are merely intended to direct the attention of geometers to the possibility of further investigations.

Chapter V considers the application of the theory of harmonic integrals to certain problems in the theory on continuous groups. The reader will require some slight knowledge of group theory in this chapter, but, following the precedent of Chapter I, I have begun the chapter with a brief summary of the results which will be used. The chapter shows that our theory provides an alternative method of considering the invariant integrals introduced by Cartan in the topological theory of groups. In a number of important cases the results obtained coincide exactly with those found by Cartan. The chapter concludes with the determination of the Betti numbers of the group manifolds associated with the four main classes of simple groups. In this I follow closely the work of Brauer and Weyl, though in places it is modified by the use of properties of harmonic integrals.

In the earlier stages of preparing this book I had the advantage of much useful criticism from Dr J. H. C. Whitehead, of Balliol College, Oxford, but after the outbreak of war it was not possible for me to continue receiving the benefit of his advice. Prof. T. A. A. Broadbent, of the Royal Naval College, Greenwich, has read the manuscript, and has helped greatly in reading the proofs. I wish to express my thanks to both of these gentlemen for their great assistance, and to the staff of the Cambridge University Press for the care which they have taken in the printing of this book.

W. V. D. H.

Pembroke College, Cambridge
September 1940