Chapter I

RIEMANNIAN MANIFOLDS

1·1. **Introduction.** The theory of harmonic integrals has its origin in an attempt to generalise the well-known existence theorem of Riemann for the everywhere finite integrals on a Riemann surface. In making the generalisation, the first necessity is to determine the nature of the n-dimensional space which is to play the part of the Riemann surface. The space which we obtain is called an n-dimensional Riemannian manifold. A Riemannian manifold of two dimensions is not, however, the same as a Riemann surface, and, as an introduction to the ideas with which we shall deal, we shall first consider the difference between the two concepts. These considerations will lead up to, and may help to elucidate, the formal definition of Riemannian manifolds of n dimensions which will be given later.

Let us construct in the usual way the Riemann surface for the algebraic equation, over the field of complex numbers,

$$F(z, w) = 0,$$

which defines w as an algebraic function of the variable z. This Riemann surface is a closed, orientable (i.e. two-sided) surface on which we introduce certain local coordinate systems, which we call allowable coordinate systems. In the neighbourhood of a place on the surface for which $z = a$ we take as one allowable coordinate system (σ, τ), where

$$z-a = t = \sigma+i\tau, \quad \text{or} \quad z-a = t^n = (\sigma+i\tau)^n,$$

according as the place is the origin of a linear branch, or a branch of order n. If z is infinite at the place, we replace $z-a$ by z^{-1}. Then (x_1, x_2) are allowable coordinates in the neighbourhood of the place if

$$y = x_1+ix_2 = f(t)$$

is an analytic function of t which is simple (*schlicht*) in the neighbourhood of the origin. All the allowable coordinate systems are obtained in this way, and we may call y an allowable complex parameter in the neighbourhood of the place.

If (x_1, x_2) and (x'_1, x'_2) are two allowable coordinate systems in the neighbourhood of a place, x'_1 and x'_2 are analytic functions of (x_1, x_2), which satisfy certain equations (the Cauchy-Riemann equations), and there exists a relation

$$(dx'_1)^2 + (dx'_2)^2 = \lambda[(dx_1)^2 + (dx_2)^2]$$

between the differentials at a point, where λ is a positive analytic function. Thus by means of the allowable coordinate systems we define a local geometry on the Riemann surface which is known as *conformal geometry*. In this geometry, length has no significance, but angles can be defined.

If we make a birational transformation of the fundamental algebraic equation, we obtain a new equation and corresponding to it a new Riemann surface. We denote the original Riemann surface by R and the new one by R'. There is a (1-1) continuous correspondence between the points of R and R', that is, R and R' are homeomorphic. The homeomorphism has, however, some special properties. If P and P' are corresponding points of R and R', and (x_1, x_2) and (x'_1, x'_2) are allowable local coordinates valid in their neighbourhoods, the equations giving the homeomorphism in the neighbourhoods of P and P' are

$$x'_i = f'_i(x_1, x_2) \qquad (i = 1, 2),$$
$$x_i = f_i(x'_1, x'_2) \qquad (i = 1, 2),$$

where the functions are analytic. This homeomorphism is therefore an *analytic homeomorphism*. Moreover,

$$x'_1 + ix'_2 = X'(x_1 + ix_2),$$
$$x_1 + ix_2 = X(x'_1 + ix'_2),$$

where the functions are analytic, and hence

$$(dx_1)^2 + (dx_2)^2 = \mu[(dx'_1)^2 + (dx'_2)^2],$$

where μ is a positive function. Thus the conformal properties of R and R' are preserved in the homeomorphism. The homeomorphism is therefore a conformal representation of the one surface on the other. Conversely, if two Riemann surfaces are conformally representable on one another by an analytic homeomorphism, the algebraic equations to which they correspond are birationally equivalent. The surfaces can therefore be regarded as equivalent.

1·2. The features of a Riemann surface which we wish to emphasise are that it is a closed orientable surface carrying certain allowable coordinate systems which specify a local geometry, and that between equivalent Riemann surfaces there is a homeomorphism which relates the local geometries of the surfaces. A Riemannian manifold of two dimensions is also a closed orientable surface, but differs from a Riemann surface in the systems of coordinates which are allowable, and in the local geometry.

The allowable coordinate systems on a Riemannian manifold of two dimensions are characterised by the properties (*a*) that there is at least one allowable system valid in the neighbourhood of any point, and (*b*) that, if (x_1, x_2) are allowable coordinates in a neighbourhood N, and x'_1 and x'_2 are differentiable functions of (x_1, x_2) in N, the necessary and sufficient conditions that (x'_1, x'_2) are allowable coordinates in N are:

(i) the Jacobian $$\frac{\partial(x'_1, x'_2)}{\partial(x_1, x_2)}$$

is different from zero in N and

(ii) (x'_1, x'_2) do not assume the same set of values at two distinct points of N. If the functions x'_1, x'_2 have continuous derivatives of order u, where u is a positive integer or zero, they are said to be *of class u* (class ω if they are analytic); and if the equations of transformation relating any two allowable systems of coordinates are of class u, the manifold is said to be *of class u*.

While the local geometry of a Riemann surface is conformal geometry, the local geometry on a Riemannian manifold of two dimensions is *Riemannian geometry*. We associate with each allowable coordinate system (x_1, x_2) a positive definite quadratic differential form

$$E\,dx_1^2 + 2F\,dx_1\,dx_2 + G\,dx_2^2,$$

where E, F, G are functions of (x_1, x_2), and are of class $(u-1)$ if the manifold is of class u. If

$$E'dx_1'^2 + 2F'dx_1'dx_2' + G'dx_2'^2$$

is the differential form associated with another coordinate system (x_1', x_2') valid in the same neighbourhood, the coefficients of the two expressions are connected by the equations

$$\begin{aligned} E' &= E\left(\frac{\partial x_1}{\partial x_1'}\right)^2 + 2F\frac{\partial x_1}{\partial x_1'}\frac{\partial x_2}{\partial x_1'} + G\left(\frac{\partial x_2}{\partial x_1'}\right)^2, \\ F' &= E\frac{\partial x_1}{\partial x_1'}\frac{\partial x_1}{\partial x_2'} + F\left(\frac{\partial x_1}{\partial x_1'}\frac{\partial x_2}{\partial x_2'} + \frac{\partial x_1}{\partial x_2'}\frac{\partial x_2}{\partial x_1'}\right) + G\frac{\partial x_2}{\partial x_1'}\frac{\partial x_2}{\partial x_2'}, \\ G' &= E\left(\frac{\partial x_1}{\partial x_2'}\right)^2 + 2F\frac{\partial x_1}{\partial x_2'}\frac{\partial x_2}{\partial x_2'} + G\left(\frac{\partial x_2}{\partial x_2'}\right)^2. \end{aligned}$$

The quadratic differential forms enable us to define in an invariant way the length of any arc on the manifold, as in elementary differential geometry.

Two Riemannian manifolds of two dimensions which are of class u are equivalent if there exists a (1-1) correspondence between their points which is given locally in terms of allowable coordinate systems by equations of class u, and is such that the lengths of corresponding arcs are the same. If two Riemannian manifolds of two dimensions are in (1-1) correspondence of class u, we can always choose allowable coordinate systems in the neighbourhoods of any pair of corresponding points so that in these neighbourhoods corresponding points have the same coordinates. The necessary and sufficient condition that the manifolds should be equivalent is that in these coordinate systems the coefficients of the quadratic differential forms on the two manifolds should be equal at corresponding points. If the manifolds are not equivalent, it may yet happen that in these coordinate systems the coefficients of the quadratic differential forms are proportional at corresponding points. In this case we say that the manifolds are *conformally related* by the homeomorphism.

1·3. There is clearly a considerable difference in character between a Riemann surface and a Riemannian manifold of two dimensions. It is therefore worth while showing how we can pass from the one to the other, and how we can obtain properties of a Riemann surface from a knowledge of the properties of a Riemannian manifold of two dimensions. Among the allowable systems of coordinates on a Riemannian manifold we can find a sub-set G with the property that the equations of transformation from one coordinate system of G to another are analytic. In G there is a sub-set G_1 for which the fundamental quadratic form is given by

$$\lambda\,(dx_1^2+dx_2^2).$$

If (x_1, x_2) and (x_1', x_2') are two coordinate systems of G_1 valid in the same region, it is well known that

$$x_1'+ix_2' = f(x_1 \pm ix_2),$$

where the function is analytic. There is a sub-set G_2 of G_1 for which the upper sign holds. The closed orientable surface which is the Riemannian manifold, and which has G_2 as the set of allowable coordinate systems, is a Riemann surface. Riemannian manifolds of two dimensions which are conformally homeomorphic define in this way equivalent Riemannian surfaces. In a later chapter we shall see that, conversely, a Riemann surface defines, in a unique way, an infinite set of Riemannian manifolds of two dimensions, any two of which are conformally homeomorphic. Certain invariants, in particular those which we shall call harmonic integrals, of a Riemannian manifold are unaltered when we pass from one manifold to a conformally homeomorphic manifold, and therefore define invariants of a Riemann surface. While this method of obtaining invariants of a Riemann surface is extremely artificial, the generalisation of it enables us to obtain invariants of the Riemannian of an algebraic variety of any number of dimensions which have not as yet been obtained by other means. But there are also other fields in which we can apply the theory of Riemannian manifolds.

2·1. **Manifolds of class u.** We now define a Riemannian manifold of dimension n. These come within the category of manifolds of class u, as defined by Veblen and Whitehead [10]†, and the reader is referred to their tract for a more elaborate examination of the structure of such manifolds than we give here. We shall give only a brief description of their character.

In order to define any space, we begin with a set of undefined elements which we call *points*, and impose certain conditions on them. We postulate the existence of certain sub-sets, which we call *neighbourhoods*, and we suppose that every point lies in at least one neighbourhood. We call the neighbourhoods which contain a point the neighbourhoods of the point, and when we say that a property holds in the neighbourhood of a point we mean that there is at least one neighbourhood of the point in which the property holds.

The neighbourhoods with which we shall deal are also assumed to have the properties:

(*a*) if any two neighbourhoods N and N' have a point in common, there is a neighbourhood of this point which lies in both N and N';

(*b*) if P and Q are distinct points of the set, there exist neighbourhoods of P and Q which have no point in common.

A set of points and a set of neighbourhoods with these properties form a *space*. By putting further restrictions on the neighbourhoods we can determine different types of space.

† References given in this way relate to the list of references at the end of each chapter.

The space which we now define is called a *manifold*. We require first that the points of any neighbourhood shall be in (1-1) correspondence with the set N of points of the real number space $(x_1, \ldots, x_n)$ of n dimensions which satisfy the inequalities

$$|x_i| < \delta \qquad (i = 1, \ldots, n)$$

where δ is a given positive number. If $(\bar{x}_1, \ldots, \bar{x}_n)$ is any point of N, there exists a number δ_1 such that all the points given by

$$|x_i - \bar{x}_i| < \delta_1 \qquad (i = 1, \ldots, n)$$

lie in N, and for any such δ_1 we suppose that those points of the space which correspond to the points of N satisfying

$$|x_i - \bar{x}_i| < \delta_1 \qquad (i = 1, \ldots, n)$$

form a neighbourhood.

The number n is the same for all neighbourhoods, and is called the *dimension* of the manifold.

The correspondence between a neighbourhood of a manifold and the points of the number space given by

$$|x_i| < \delta \qquad (i = 1, \ldots, n)$$

defines a *coordinate system* in the neighbourhood. Since any point lies in several neighbourhoods, there exist several coordinate systems valid at a point. Any two which are valid at the same point are both valid in some neighbourhood of the point. Let $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ be two coordinate systems valid in a neighbourhood N. Then, at points of N there is a (1-1) correspondence between the two systems of coordinates which can be expressed by the equations

$$x_i = f_i(x'_1, \ldots, x'_n) \qquad (i = 1, \ldots, n),$$

$$x'_i = g_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n).$$

Such a system of equations defines a transformation of coordinates. By saying that the manifold is of class u, we mean that the neighbourhoods are so restricted that the functions $f_i(x'_1, \ldots, x'_n)$, $g_i(x_1, \ldots, x_n)$, for $i = 1, \ldots, n$, are of class u, and that, when $u > 0$, the equations of transformation are *regular* in N, that is, that the Jacobians

$$\left|\frac{\partial f_i}{\partial x'_j}\right|, \quad \left|\frac{\partial g_i}{\partial x_j}\right|$$

are different from zero at all points of N. The ideas of *limit point*, etc. on the manifold can be defined in terms of these coordinate systems.

It is convenient to admit further allowable coordinate systems in a neighbourhood N. Let $y_1, \ldots, y_n$ be real functions of $(x_1, \ldots, x_n)$ of class u, where $(x_1, \ldots, x_n)$ are coordinates of the type already defined, and suppose that the functions are defined at all points of N and satisfy the conditions:

(i) $\left|\frac{\partial y_i}{\partial x_j}\right|$ is different from zero in N (when $u > 0$); and

(ii) there exists no pair of points x and x' in N for which

$$y_i(x) = y_i(x') \qquad (i = 1, \ldots, n).$$

Each point of N can then be identified by the set of values of the functions $y_1, \ldots, y_n$ at the point. We shall therefore admit $(y_1, \ldots, y_n)$ as allowable coordinates in N.

We now impose two conditions on the neighbourhoods which restrict the nature of the manifold as a whole. First, the manifold must be *finite*; that is, there must be a finite number of neighbourhoods $N_1, \ldots, N_r$ whose sum entirely covers the manifold. Secondly, the manifold must be *connected*; that is, the neighbourhoods $N_1, \ldots, N_r$ cannot be divided into sets, each non-vacuous, $N_{i_1}, \ldots, N_{i_s}$ and $N_{i_{s+1}}, \ldots, N_{i_r}$, which have no point in common.

2·2. A manifold, as we have defined it, can be represented in a simple manner as a locus in Euclidean space. In an n-dimensional number space $(x_1, \ldots, x_n)$, any set of points in (1-1) continuous correspondence with the points interior to a sphere of the space is called a *simplicial region* of the space. We consider a set of points in the Euclidean space $(X_1, \ldots, X_N)$ which (i) lies entirely in a finite region of the space; (ii) forms a connected set; (iii) has the property that those points of the set which lie within a distance δ of any given point P of the set are in (1-1) correspondence of class u with the points of a simplicial region N in $(x_1, \ldots, x_n)$, and are given by equations

$$X_i = f_i(x_1, \ldots, x_n) \qquad (i=1, \ldots, N),$$

where the functions are of class u, and where the matrix

$$\left(\frac{\partial X_i}{\partial x_j}\right)$$

is of rank n at all points of N. Such a set of points is said to form a *locus of class u* in $(X_1, \ldots, X_N)$. We now propose to show that any manifold of class u can be represented as a locus of class u_1 in a Euclidean space, where u_1 is any finite integer not exceeding u.

To prove this, we consider the neighbourhoods $N_1, \ldots, N_r$ whose sum covers the manifold. Let $(x_1^j, \ldots, x_n^j)$ be a coordinate system in N_j obtained by representing N_j on

$$|x_i^j| < \delta \qquad (i=1, \ldots, n)$$

in the number space $(x_1^j, \ldots, x_n^j)$. We define $2nr$ functions y_i^j, z_i^j of the points of the manifold as follows. The functions y_i^j, z_i^j are zero at all points not in N_j. At points of N_j,

$$y_i^j = [(x_i^j)^2 - \delta^2]^s,$$
$$z_i^j = x_i^j y_i^j.$$

If $s > u_1$, these functions, regarded as functions of allowable coordinates valid in the neighbourhood of any point of the manifold, are functions of class u_1. A point P of the manifold lies in at least one neighbourhood N_j, and for this j, y_i^j is not zero at P. Let P and Q be two distinct points of the manifold. If P lies in N_j and Q does not lie in N_j, then

$$y_i^j(P) \neq 0, \quad y_i^j(Q) = 0.$$

If both P and Q lie in N_j, and if

$$y_i^j(P) = y_i^j(Q), \quad z_i^j(P) = z_i^j(Q),$$

then

$$x_i^j(P) = x_i^j(Q).$$

Thus we can always find i, j so that either

$$y_i^j(P) \neq y_i^j(Q),$$

or

$$z_i^j(P) \neq z_i^j(Q).$$

Now let $(X_1, \ldots, X_{2nr})$ be coordinates in a Euclidean space of $2nr$ dimensions. Consider the locus defined by

$$X_{(j-1)n+i} = y_i^j,$$

$$X_{(r+j-1)n+i} = z_i^j,$$

for $i = 1, \ldots, n$; $j = 1, \ldots, r$, where the argument of the functions on the right-hand sides is a point P which ranges over the manifold. This locus is in (1-1) correspondence with the manifold. It is easily verified that it is a locus of class u_1, as defined above. Conversely, we may show that a locus of class u_1 is a manifold of class u_1, the neighbourhoods being defined in the obvious way.

A representation of a manifold of class u as a locus of class u_1 in a Euclidean space will be referred to as a Euclidean representation, and will be found useful later in proving some general theorems. We observe that if u is finite we may take $u_1 = u$, but if $u = \omega$ this is not possible (but see Whitney [13]).

2·3. At this stage we may interpolate a remark concerning the class number u. We shall not be particularly interested in proving our results under a minimum number of conditions imposed on the manifold, and for the applications which we shall make it will only be necessary to suppose that u is sufficiently large for the operations which we perform. In fact, it will appear that every result which we establish will be valid if $u > 6$, and many will be valid for smaller values of u. We shall often leave as an exercise to the reader the problem of deter-