INEQUALITIES

Oh! the little more, and how much it is!
And the little less, and what worlds away!

ROBERT BROWNING

INEQUALITIES

By

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PREFACE TO FIRST EDITION

This book was planned and begun in 1929. Our original intention was that it should be one of the *Cambridge Tracts*, but it soon became plain that a tract would be much too short for our purpose.

Our objects in writing the book are explained sufficiently in the introductory chapter, but we add a note here about history and bibliography. Historical and bibliographical questions are particularly troublesome in a subject like this, which has applications in every part of mathematics but has never been developed systematically.

It is often really difficult to trace the origin of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; and no accessible statement of it may be quite complete. We have almost always found, even with the most famous inequalities, that we have a little new to add.

We have done our best to be accurate and have given all references we can, but we have never undertaken systematic bibliographical research. We follow the common practice, when a particular inequality is habitually associated with a particular mathematician's name; we speak of the inequalities of Schwarz, Hölder, and Jensen, though all these inequalities can be traced further back; and we do not enumerate explicitly all the minor additions which are necessary for absolute completeness.

We have received a great deal of assistance from friends. Messrs G. A. Bliss, L. S. Bosanquet, R. Courant, B. Jessen, V. Levin, R. Rado, I. Schur, L. C. Young, and A. Zygmund have all helped us with criticisms or original contributions. Dr Bosanquet, Dr Jessen, and Prof. Zygmund have read the

proofs, and corrected many inaccuracies. In particular, Chapter III has been very largely rewritten as the result of Dr Jessen's suggestions. We hope that the book may now be reasonably free from error, in spite of the mass of detail which it contains.

Dr Levin composed the bibliography. This contains all the books and memoirs which are referred to in the text, directly or by implication, but does not go beyond them.

G. H. H.
J. E. L.
G. P.

Cambridge and Zürich
July 1934

PREFACE TO SECOND EDITION

The text of the first edition is reprinted with a few minor changes; three appendices are added.

J. E. L.
G. P.

Cambridge and Stanford
March 1951

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