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1 Basic concepts

1.1 Introduction

This chapter is the introduction to structures and designs and, while it is completely elementary, it is essential to the rest of the book. Section 1.2 contains the basic definitions. In Section 1.3 we then give a number of examples. We begin by listing some small carefully chosen ones to illustrate the meanings of the earlier definitions but then go on to examples based on projective and affine geometry. Obviously knowledge of classical geometry will help the reader to follow and understand these examples, but we have tried to make our explanations as full as possible and to make the entire chapter self-contained. Nevertheless the importance of finite projective and affine geometry to design theory cannot be overemphasised and we include some excellent references for further reading. In Section 1.4 we return to definitions and results about arbitrary structures, in particular relating a structure to others which can be constructed from it or from which it can be constructed. Section 1.5 studies the incidence matrix of a structure, already introduced in Section 1.2, and uses it to prove a number of basic theorems: Fisher's Inequality and properties of square structures and symmetric designs in particular. Polarities are introduced and the incidence matrix is exploited to prove a number of their basic and important properties. In Section 1.6 the notion of tactical decomposition of a structure is introduced, Block's Lemma is proved, and applications to automorphism groups (in particular the Orbit Theorems) are deduced. Resolutions and parallelisms are briefly introduced as well. Section 1.7 contains a brief discussion of graph theory and some of its connections with the theory of structures and designs.

The reader should bear in mind that most of the concepts in this chapter are extremely important, not only in the rest of the book, but also in the literature. The 'small' examples in Section 1.3 are mainly pedagogical, but the projective and affine geometries also introduced in that section occur again and again in design theory. The notion of a restriction or residue in Section 1.4 is also very important. The incidence matrix is central to some of the most important theorems in the subject (some of which, like Fisher's Inequality and the results about symmetric designs, are proved in Section 1.5). Automorphism groups, while they are only introduced in a very general and elementary way in this chapter, are of the greatest importance in the entire subject, and arise repeatedly throughout the book.

1.2 Basic definitions and properties

A *structure* (or sometimes an *incidence structure*) is two finite sets of objects, called *points* and *blocks*, with an *incidence relation* $\mathscr{I}$ between them. Usually we shall use upper case Latin letters to denote points and lower case Latin letters to denote blocks. Given any block there is a set of points incident with it, and we shall frequently find it convenient to identify the block with this point set. Thus if a point P is incident with a block y, we can write either $P\mathscr{I}y$ or $P \in y$, and we will use any convenient expression such as 'P is on y', 'y contains P', 'y is on P', etc. An equivalent approach is to regard a structure as a finite point set $\mathscr{P}$, a finite block set $\mathscr{B}$, and the incidence given by a subset $\mathscr{I} \subseteq \mathscr{P} \times \mathscr{B}$; then a point P is on the block y if and only if $(P, y) \in \mathscr{I}$. In this case the structure is denoted by $(\mathscr{P}, \mathscr{B}, \mathscr{I})$.

If P is a point and y is a block of a structure $\mathscr{S}$, then $\langle P \rangle$ and $\langle y \rangle$ are often used to denote the set of blocks on P and the set of points on y, respectively. As usual, we write $|\mathscr{U}|$ for the number of elements in a set $\mathscr{U}$, but we will write $|P|$ for $|\langle P \rangle|$ and $|y|$ for $|\langle y \rangle|$.

Clearly these definitions could be extended to 'infinite structures', but much of the interesting theory and applications is different for the finite and infinite cases. We restrict our study to the finite case, though we may from time to time point out possible extensions to the infinite case.

If $\mathscr{S}$ is a structure then we usually denote the number of points and blocks of $\mathscr{S}$ by v and b respectively. Clearly, for a structure to be interesting it should have non-empty sets of both points and blocks, and we therefore assume that any structure has $v \neq 0$, $b \neq 0$, unless the contrary is explicitly stated or allowed.

When we defined a structure above we said that it would often be convenient to identify a block with the set of points on it. If we do this, however, there is no reason why two distinct blocks might not be identified with the same set of points. Indeed, this will often occur, and when it does we say that the structure has *repeated blocks*. However, if $\mathscr{S}$ has repeated blocks, we can construct from $\mathscr{S}$ a new structure which does not, simply by 'deleting' all but one of a set of blocks which are incident with the same set of points. More formally we can define an equivalence relation R on the set of blocks by: xRy if $\langle x\rangle = \langle y\rangle$. The *multiplicity* of the block x is the size of the R-equivalence class of x, and if x has multiplicity greater than 1, then we say that x is a *repeated* block. Then we define a new structure, the *reduction* of $\mathscr{S}$ (or '$\mathscr{S}$ reduced'), written $\mathscr{S}/R$, whose points are the points of $\mathscr{S}$ and whose blocks are the R-equivalence classes of blocks of $\mathscr{S}$, with P on the equivalence class of x if P is on x in $\mathscr{S}$. Most of the time we shall be interested only in reduced structures, although frequently an important part of a proof, say, will consist of showing that a structure has no repeated blocks.

If an element of a structure $\mathscr{S}$ is incident with 0 or 1 other elements of $\mathscr{S}$, then we call this element *isolated*; on the other hand, a *full* element is one which is incident with every other possible element (i.e. point with blocks or block with points). We can successively remove all full elements, then all isolated elements, then all full elements, etc., to reach a new structure $\bar{\mathscr{S}}$ (which might be empty!), called '$\mathscr{S}$ *standardised*'. Finally, a structure is *totally reduced* if it is both standardised and reduced: that is, it has no full or isolated elements, and no repeated blocks. These concepts, 'isolated', 'full', 'standardised' and 'totally reduced' are usually of very little relevance, and in this book will only arise in dealing with some examples in early chapters.

Suppose that $\mathscr{S}$ is a structure with v points, b blocks, where $v>0$ and $b>0$, and that the points of $\mathscr{S}$ are indexed $P_1, P_2, \ldots, P_v$, while the blocks are $x_1, x_2, \ldots, x_b$. Then an *incidence matrix* $A=(a_{ij})$ for $\mathscr{S}$ is a v by b matrix where $a_{ij}=1$ if P_i is on x_j and $a_{ij}=0$ otherwise. Obviously all the information about $\mathscr{S}$ (up to 'isomorphism', a term to be defined later) is contained in A. (A matrix is called a *(0, 1)-matrix* if all its entries are 0 or 1; clearly any incidence matrix is a (0, 1)-matrix.) It is important to note that A is certainly not uniquely determined by $\mathscr{S}$ since it depends on the indexing of the points and blocks. Different indexings give rise to different incidence matrices. However, as one might expect, there is a close relation between the incidence matrices for a given structure $\mathscr{S}$. Suppose that A is the matrix of $\mathscr{S}$ given by an indexing $P_1, \ldots, P_v$ and $x_1, \ldots, x_b$ and that B is the matrix given by $Q_1, \ldots, Q_v$ and $x_1, \ldots, x_b$ (i.e. the points are labelled differently but the blocks are the same). Then each Q_i is one of the P_j so we get a permutation θ on $\{1, \ldots, v\}$ given by $i^\theta = j$ if and only if $Q_i = P_j$. But this means that, for any i, row i of B is row i^θ of A. In other words, B is obtained from A by permuting its rows. This argument can be extended to solve the following:

Exercise 1.1. If A and B are two incidence matrices for a structure $\mathscr{S}$, show that there exist permutation matrices P and Q such that $PAQ=B$.

Thus if $\mathscr{S}$ is a structure then any two incidence matrices for $\mathscr{S}$ are equivalent and, consequently, have the same rank. This enables us to define the *rank* of a structure $\mathscr{S}$ to be the rank of one (and hence all) of its incidence matrices (over the field of rationals).

If $\mathscr{S}$ and $\mathscr{T}$ are two structures we define an *isomorphism* α from $\mathscr{S}$ onto $\mathscr{T}$ to be a one-to-one mapping from the points of $\mathscr{S}$ onto the points of $\mathscr{T}$ and the blocks of $\mathscr{S}$ onto the blocks of $\mathscr{T}$ such that P is on x if and only if P^α is on x^α. If there is an isomorphism from $\mathscr{S}$ onto $\mathscr{T}$ then we say that $\mathscr{S}$ and $\mathscr{T}$ are isomorphic and write $\mathscr{S} \cong \mathscr{T}$. Clearly, as always, isomorphism is an equivalence relation.

Lemma 1.1. Any (0, 1)-matrix is an incidence matrix of a structure. If A and B are (0, 1)-matrices then they are incidence matrices for isomorphic structures if and only if there exist permutation matrices P, Q with $PAQ = B$.

Proof. If A is a v by b (0, 1)-matrix then we define a set of points $P_1, \ldots, P_v$ and a set of blocks $x_1, \ldots, x_b$, where P_i is incident with x_j if and only if the (i, j) entry of A is 1. This is clearly a structure.

Suppose that A is the incidence matrix of a structure $\mathscr{S}$ with points $P_1, \ldots, P_v$ and blocks $x_1, \ldots, x_b$ such that $a_{ij} = 1$ if and only if P_i is on x_j. If B is an incidence matrix for a structure $\mathscr{T}$ which is isomorphic to $\mathscr{S}$ than, clearly, B is also a v by b matrix. Let $Q_1, \ldots, Q_v$ and $y_1, \ldots, y_b$ be the labelling such that $b_{ij} = 1$ if and only if Q_i is on y_j. If we label the elements of $\mathscr{T}$ as $R_1, \ldots, R_v, z_1, \ldots, z_b$, where $R_i = P_i^\alpha$, $z_j = x_j^\alpha$ $(1 \leqslant i \leqslant v,\ 1 \leqslant j \leqslant b)$, then, since α is an isomorphism, $a_{ij} = 1$ if and only if R_i is on z_j. Thus A is an incidence matrix for $\mathscr{T}$ and, by Exercise 1.1, there exist permutation matrices P, Q with $PAQ = B$.

Now suppose that A, B are v by b (0, 1)-matrices such that there exist permutation matrices P, Q with $PAQ = B$. Let θ be the permutation on $\{1, \ldots, v\}$ and let ϕ be the permutation on $\{1, \ldots, b\}$ given by $a_{i^\theta j^\phi} = b_{ij}$ (i.e. θ is determined by P and ϕ by Q). Let $\mathscr{S}$ be the incidence structure such that it has A as an incidence matrix when its elements are labelled $P_1, \ldots, P_v, x_1, \ldots, x_b$ and let $\mathscr{T}$ have B as an incidence matrix when its elements are labelled $Q_1, \ldots, Q_v, y_1, \ldots, y_b$. Now define α from $\mathscr{S}$ onto $\mathscr{T}$ by $P_i^\alpha = Q_j$, where $j^\theta = i$ and $x_l^\alpha = y_m$, where $m^\phi = l$. Thus $\alpha: P_{j^\theta} \to Q_j$ and $x_{m^\phi} \to y_m$. Since α is clearly a bijection (one-to-one and onto mapping) from the points and blocks of $\mathscr{S}$ onto the points and blocks of $\mathscr{T}$, in order to show that α is an automorphism we have only to show the P_i^α is on x_l^α if and only if P_i is on x_i, i.e. that P_{j^θ} is on x_{m^ϕ} if and only if Q_j is on y_m. But P_{j^θ} is on x_{m^ϕ} whenever $a_{j^\theta m^\phi} = 1$ and Q_j is on y_m if $b_{jm} = 1$. But, from the definitions on θ and ϕ, $a_{j^\theta m^\phi} = b_{jm}$ and the lemma is proved. □

This lemma illustrates what an exceedingly simple object a structure is. It is, in fact, too simple to be of any interest by itself, but as soon as we begin to impose extra conditions then it becomes both interesting and useful.

Note, by the way, that if A is the incidence matrix of a structure $\mathcal{S}$ given by the indexing $P_1, \ldots, P_v, x_1, \ldots, x_b$ then the zero entries in row i are in those columns which represent the blocks which are not incident with P_i. Thus an isolated element will lead to a row (or column) of A with at most one non-zero entry. Similarly, a full element is equivalent to a row (or column) where every entry is 1. Finally we observe again that two identical columns in A means that $\mathcal{S}$ has a repeated block. Thus a (0, 1)-matrix A is the incidence matrix of a totally reduced structure if

(i) each row and column has at least one zero entry and at least two ones, and

(ii) A has no two identical columns.

We now introduce the first, and most common, condition which we impose on our structures.

A structure $\mathcal{S}$ will be called *uniform* if its block set is non-empty and if each of its blocks contains exactly $k>0$ points. The uniformity of a structure can be recognised from its incidence matrix in a very simple way.

For any positive integers m, n we denote by $J_{m,n}$ the m by n matrix with all its entries 1; we write J_m for $J_{m,m}$ and $\mathbf{j}_m$ for $J_{1,m}$, so that $\mathbf{j}_m$ is a row vector. If the context is clear, we sometimes write J or $\mathbf{j}$ for $J_{m,n}$ or $\mathbf{j}_m$.

Exercise 1.2. If A is an incidence matrix for a structure $\mathcal{S}$, show that

(a) $J_{m,v}A$ is an m by b matrix whose ith column consists completely of the entry $|y_i|$ (where y_i is the ith block of $\mathcal{S}$);

(b) $\mathcal{S}$ is uniform if and only if $J_{m,v}A = kJ_{m,b}$ for a constant $k \neq 0$.

If we now define a structure to be *regular* if $|P| = r > 0$ for all points P, then Exercise 1.2 has an obvious extension: $\mathcal{S}$ is regular if and only if $AJ_{b,m} = rJ_{v,m}$ for a constant $r \neq 0$.

Let $\mathcal{S}$ be a structure with $v>0$ points. If there exist integers λ, t with $0<\lambda$ and $0 \leqslant t \leqslant v$ such that every subset of t points of $\mathcal{S}$ is incident with exactly λ common blocks then we say that $\mathcal{S}$ is a *t-structure for λ*, or merely a *t-structure*. (Note that a 0-structure is merely a non-empty structure.) A uniform t-structure with block size k is called a (uniform) *t-(v, k, λ) structure* or a *t-structure for* (v, k, λ), where v is the number of points and λ is the number of common blocks on t points. Note that, by definition, any uniform structure or t-structure must be non-empty. Also by talking about a t-structure for (v, k, λ)

we are automatically implying that it is uniform by referring to the 'k'. Hence we shall usually drop the word 'uniform' in this situation.

Uniform t-structures have the interesting, and very useful, property that they are also uniform s-structures for all s satisfying $0 \leqslant s \leqslant t$. We shall often refer to a set of j symbols as a j-set.

Theorem 1.2. If $\mathscr{S}$ is a t-structure for (v, k, λ) then, for any integer s satisfying $0 \leqslant s < t$, there are exactly λ_s blocks of $\mathscr{S}$ which are incident with any given s-set of points of $\mathscr{S}$, where

$$\lambda_s = \lambda \frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)}.$$

Proof. If $\mathscr{B}$ is any fixed subset of s points of $\mathscr{S}$, let m be the number of blocks of $\mathscr{S}$ which contain $\mathscr{B}$. We shall prove the theorem by computing m and showing that it depends only on s and is independent of the choice of $\mathscr{B}$. In order to compute m we define an *admissible pair* $(\mathscr{T}, y)$ to be a set $\mathscr{T}$ of t points which contains $\mathscr{B}$ and a block y which contains $\mathscr{T}$, and then compute the number of admissible pairs in two different ways.

Each of the m blocks which contain $\mathscr{B}$ contains $\binom{k-s}{t-s}$ t-sets which contain $\mathscr{B}$, so the number of admissible pairs is $m\binom{k-s}{t-s}$. On the other hand, there are $\binom{v-s}{t-s}$ ways to choose a set $\mathscr{T}$ of t points which contains $\mathscr{B}$, and each of these sets is on exactly λ blocks. Thus the number of admissible pairs is also $\lambda\binom{v-s}{t-s}$. Equating these two values for the number of admissible pairs, we see that

$$m = \lambda \frac{(v-s)\cdots(v-t+1)}{(k-s)\cdots(k-t+1)}$$

and the theorem is proved. □

Corollary 1.3. If $\mathscr{S}$ is a t-structure for (v, k, λ), $t > 0$, and s is any integer satisfying $0 < s < t$, then $\mathscr{S}$ is an s-structure for (v, k, λ_s), and, in particular, $\mathscr{S}$ is regular.

Proof. Obvious. □

One immediate consequence of Theorem 1.2 is the non-existence of uniform t-structures for most choices of v, k, λ. This is because each of the quantities λ_s

$(0 \leqslant s \leqslant t)$, where $\lambda_t = \lambda$, must be an integer. As a simple illustration the reader should solve the following exercise.

Exercise 1.3

(a) Show that there cannot exist a 4-structure for $(11, 7, 2)$.

(b) Show that if $t=5$, $v=24$, $k=8$ and $\lambda=1$ then λ_4, λ_3, λ_2 and λ_1 are all integers.

If $\mathscr{S}$ is a t-(v, k, λ) structure then, as always, we will denote the number of blocks by b. If $t \geqslant 1$ then, by Theorem 1.2, the number of blocks through a point is a constant which we denote by r, and if $t \geqslant 2$ then the integer $n = r - \lambda_2$ is called the *order* of $\mathscr{S}$. (Note that $b = \lambda_0$, $r = \lambda_1$, and $\lambda = \lambda_t$.) The integers $t, v, b, k, \lambda, r, n$ are called the *parameters* of $\mathscr{S}$. Clearly, they are not independent and we now restate certain special cases of Theorem 1.2 to emphasise some of the most important relations between them.

Corollary 1.4. If $\mathscr{S}$ is a t-structure for (v, k, λ) then:

(a) $$b = \lambda \frac{v(v-1)\cdots(v-t+1)}{k(k-1)\cdots(k-t+1)};$$

(b) if $t > 0$ then $bk = vr$;

(c) if $t > 1$ then $r(k-1) = \lambda_2(v-1)$. □

Exercise 1.4. Define a *flag* in a structure to be a pair (P, x), where P is a point and x is a block on P. By counting flags in two ways give a direct proof of (b) of Corollary 1.4.

Exercise 1.5. Define a *2-flag* (P, Q, x) in a structure to be a pair of distinct points P, Q and x a block incident with them both. By counting 2-flags in two ways give a direct proof of (c) of Corollary 1.4.

As we have already pointed out, any uniform t-structure with $t \geqslant 1$ is also a uniform 1-structure, which means that it is regular. The concept of *flag* introduced in Exercise 1.4 is extremely useful and arises again and again; the flag-counting of that exercise (and of Exercise 1.5) is one of the most common techniques in this book, and in the theory as a whole.

If a structure $\mathscr{S}$ has equally many points and blocks then any incidence matrix for $\mathscr{S}$ is square. For this reason we call a structure with $b = v$ a *square* structure.

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Exercise 1.6. Show that a uniform 1-structure is square if and only if $k=r$.

We are now able to define the structures which are the central topic of this book. A *design* is a uniform, reduced structure, i.e. a uniform structure with no repeated blocks. (In terms of incidence matrices this means that the columns of an incidence matrix have equal sums but are all distinct.) Thus in a design a block is uniquely determined by the points on it and we may identify the blocks with distinguished subsets of points. The definition of a *t-design* is clear: namely a t-structure which is also a design. Since a design is a uniform structure, our results on uniform structures also apply to designs with the words 'uniform structure' replaced by 'design' throughout. We now introduce an important convention: if $\mathscr{S}$ is a t-design for (v, k, λ) then we say merely that $\mathscr{S}$ is a t-(v, k, λ). The point of this is simply that we shall be concerned with designs much more often than with general structures, and we want the most economical notation for them.

We have already seen that there do not exist t-(v, k, λ)s for arbitrary choices of t, v, k, λ. However, for any given t, v, k with $0 \leqslant t \leqslant k \leqslant v$ (note that these inequalities are necessary!) there is a t-$(v, k, \binom{v-t}{k-t})$. This is obtained by calling every possible k-subset of a given v-set a block. These designs are called trivial and we shall, in general, ignore them. More generally a uniform structure with block size k is called *trivial* if every k-set of points is incident with at least one block.

Exercise 1.7. If $\mathscr{S}$ is a design with block size k show that $\mathscr{S}$ is trivial if and only if $\mathscr{S}$ is a t-design for all t satisfying $0 \leqslant t \leqslant k$. (Note that a trivial uniform structure need not be a t-structure for any $t > 0$.)

We are now able to illustrate the vast difference between t-designs and uniform t-structures, and perhaps give some indication as to why we concentrate on designs. There are, at present, no known examples of non-trivial t-designs with $t \geqslant 7$. In Chapters 4 and 8 we construct some t-designs for $t = 3, 4$ and 5; in [8] some 6-designs have been constructed (indeed, the only known examples at present). The situation for uniform t-structures could hardly be more different: as we shall prove in our next theorem, for any given t, v, k, with $0 < t < k \leqslant v/2$ there is a value of λ such that there exists a non-trivial structure for t-(v, k, λ). The restrictions $0 < t < k$ are obviously necessary for a non-trivial t-structure, but the reason for restricting outselves to $k \leqslant v/2$ will not become apparent until we discuss complementary structures in Section 1.4. In the proof (which should be left until later if the reader finds it difficult) the structure constructed always has repeated blocks and, consequently, is never a design.

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Theorem 1.5. Given any positive integers t, v, k with $t<k\leqslant v/2$ there is a value of λ such that there exists a non-trivial t-(v, k, λ) structure.

Proof. Let $\mathscr{V}$ be a set of v elements. Let $\mathscr{X}_1, \ldots, \mathscr{X}_{\binom{v}{t}}$ be the t-subsets of $\mathscr{V}$, let $\mathscr{Y}_1, \ldots, \mathscr{Y}_{\binom{v}{k}}$ be the k-subsets and let A be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix given by $a_{ij}=1$ if $\mathscr{X}_i \subset \mathscr{Y}_j$ and $a_{ij}=0$ otherwise. Since $t<k\leqslant v/2$, $\binom{v}{k}>\binom{v}{t}$. Thus the rank of A is at most $\binom{v}{t}$ and the columns of A must be linearly dependent. (This is true over any field.) Thus, if $\mathbf{c}_i$ is the ith column of A $(1\leqslant i\leqslant\binom{v}{k})$ there exist rationals $\alpha_1, \ldots, \alpha_{\binom{v}{k}}$ such that $\alpha_j\neq 0$ for some value of j and $\sum_{i=1}^{\binom{v}{k}} \alpha_i\mathbf{c}_i=\mathbf{0}$. Hence there exist integers $n_1, \ldots, n_{\binom{v}{k}}$ (not all zero) such that $\sum_{i=1}^{\binom{v}{k}} n_i\mathbf{c}_i=\mathbf{0}$. Since each $\mathbf{c}_i$ is a vector of 0s and 1s, each coordinate of $\sum_{i=1}^{\binom{v}{k}} n_i\mathbf{c}_i$ is the sum of some of the n_i. Consequently some of the integers n_i are negative. Let $-m$ be the least of the n_i and put $m_i=n_i+m$ for $i=1, \ldots, \binom{v}{k}$. Clearly, by the choice of m, $m_i\geqslant 0$ for all i and there is an integer l such that $m_l=0$. Also, since $\sum_{i=1}^{\binom{v}{k}} n_i\mathbf{c}_i=\mathbf{0}$,

$$\sum_{i=1}^{\binom{v}{k}} m_i\mathbf{c}_i=\sum_{i=1}^{\binom{v}{k}} n_i\mathbf{c}_i+\sum_{i=1}^{\binom{v}{k}} m\mathbf{c}_i=m\sum_{i=1}^{\binom{v}{k}} \mathbf{c}_i.$$

But the jth coordinate of $\sum_{i=1}^{\binom{v}{k}} \mathbf{c}_i$ is the sum of the entries in the jth row of A. This is equal to the number of k-subsets containing $\mathbf{X}_j$ which, of course, is $\binom{v-t}{k-t}$. Thus

$$\sum_{i=1}^{\binom{v}{k}} m_i\mathbf{c}_i=m\sum_{i=1}^{\binom{v}{k}} \mathbf{c}_i=m\binom{v-t}{k-t}\mathbf{j},$$

where $\mathbf{j}$ is the vector having each entry 1.

We now define a structure $\mathscr{S}$. The points of $\mathscr{S}$ are the elements of $\mathscr{V}$ and for each i $(1\leqslant i\leqslant\binom{v}{k})$, $\mathscr{Y}_i$ is a block of $\mathscr{S}$ m_i times. Clearly, since each block is a k-subset, $\mathscr{S}$ is uniform and, since $m_l=0$, it is non-trivial. The number of blocks through the t-subset $\mathscr{X}_h$ is merely the hth coordinate of $\sum_{i=1}^{\binom{v}{k}} m_i\mathbf{c}_i$ which, as we have seen, is $m\binom{v-t}{k-t}$ for every h. Thus $\mathscr{S}$ is a non-trivial t-structure for $(v, k, m\binom{v-t}{k-t})$. □

It is clear from the proof that at least one of the m_i is greater than one, so that $\mathscr{S}$ is definitely not a design. (One of the n_i is greater than zero and so is m.) However, we can also see this from the value of λ.

Exercise 1.8. If $\mathscr{S}$ is a t-(v, k, λ) show that $\lambda\leqslant\binom{v-t}{k-t}$. Show also that $\lambda=\binom{v-t}{k-t}$ if and only if $\mathscr{S}$ is trivial.

Although we know that the structure $\mathscr{S}$ of Theorem 1.5 is never a design, it is not inconceivable that $\mathscr{S}$ reduced is. However, it seems very difficult to determine whether or not this is true. Clearly, since the α_i are not unique, there are many possibilities for the structure $\mathscr{S}$ and also, note, even for the integer λ. If it were possible to choose the α_i so that they only assumed two values, then there would only be two values for the n_i and, hence, only one non-zero value for the m_i. If we denote this value by c then every k-subset of $\mathscr{V}$ would be a block of $\mathscr{S}$ either 0 or c times. Under these very exceptional (and presumably unlikely) conditions $\mathscr{S}/R$ would be a t-design.

Exercise 1.9. If $\mathscr{S}$ is a t-structure for (v, k, λ) in which every block has multiplicity c show that $\mathscr{S}/R$ is a t-$(v, k, \lambda/c)$.

Problem 1.1. If $\mathscr{S}$ is a t-structure for (v, k, λ) what, if anything, can you say about $\mathscr{S}/R$? Either try to prove that $\mathscr{S}/R$ is a t-design or find a counter-example.

Exercise 1.10. Show that a 2-design with $v=8$, $k=3$ is trivial.

Exercise 1.11.* Find a non-trivial 2-structure with $v=8, k=3$. (This exercise is somewhat more difficult than it might appear, and the reader might find it profitable to return to it later. But, in connection with Exercise 1.10, it emphasises the difference between structures and designs.)

We conclude this section with some definitions.

If $\mathscr{S}$ and $\mathscr{T}$ are two structures an *anti-isomorphism* α from $\mathscr{S}$ to $\mathscr{T}$ is a bijection from the points of $\mathscr{S}$ onto the blocks of $\mathscr{T}$ and the blocks of $\mathscr{S}$ onto the points of $\mathscr{T}$ such that P is on x in $\mathscr{S}$ if and only if x^α is on P^α in $\mathscr{T}$. An anti-isomorphism of a structure $\mathscr{S}$ onto itself is called a *correlation* and, as always, an *automorphism* of $\mathscr{S}$ is an isomorphism of $\mathscr{S}$ onto itself. (Note that a structure $\mathscr{S}$ cannot have any correlations unless $v=b$; i.e. unless it is square.)

Exercise 1.12. If α is an anti-isomorphism from $\mathscr{S}$ onto $\mathscr{T}$ and β is an anti-isomorphism from $\mathscr{T}$ onto $\mathscr{S}$, show that $\alpha\beta$ is an automorphism of $\mathscr{S}$.

From Exercise 1.12 it follows that the product of two correlations of $\mathscr{S}$ is an