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PREFACE

The purpose of this book is to present the subject matter of (connected) linear algebraic monoids. This subject has been developed in the last several years, primarily by Lex Renner and the author. The basic results have been obtained. The subject is now ripe for new developments and applications. It is with the hope of attracting new researchers to the subject that this book is being written.

The theory of linear algebraic monoids represents a rather beautiful blend of ideas from abstract semigroup theory, algebraic geometry and the theory of linear algebraic groups. For example, one of the first results of the author has been to show that the group of units is solvable if and only if the regular $\mathcal{J}$–classes of the monoid form a relatively complemented lattice (they always form a finite lattice). Equivalently the monoid is a semilattice of archimedean semigroups. These semigroups were abstractly characterized by the author in his undergraduate days. From the viewpoint of semigroup theory, (von–Neumann) regular semigroups represent the most important class of semigroups. Group theorists are generally most interested in reductive algebraic groups. Well, there is a connection. L. Renner and the author have shown that a connected algebraic monoid M with zero is regular if and only if the group of units is reductive. In this situation, the author has shown that the Tits building of the group of units can be described as the local semilattice of partial $\mathcal{J}$–class idempotent cross–sections of the monoid. Going in the converse direction, L. Renner and the author have shown that the biordered set (in the sense of Nambooripad) E of idempotents of M is completely determined by the Tits building of G and a type map λ from the finite lattice $\mathcal{U}$ of $\mathcal{J}$–classes of M into a finite Boolean lattice (the power set of the Dynkin diagram). Another indication of the beauty of the subject is Renner's generalization to algebraic monoids of the classical Bruhat decomposition for algebraic groups. Renner obtains his decomposition by simply replacing the Weyl group in the Bruhat decomposition by a certain finite fundamental inverse semigroup. For the general linear group, the Weyl group is of course the symmetric group. For the full matrix semigroup, Renner's semigroup is the symmetric inverse semigroup.

There are strong connections between algebraic monoids and certain compactifications of semisimple algebraic groups and homogeneous spaces being studied by DeConcini and Procesi [14], [15]. In this regard the classification theorem of Renner is crucial. Let G be a reductive group with a maximal torus T. Renner establishes a correspondence between connected normal algebraic monoids M with zero having G as the group of units and normal torus embeddings $T \hookrightarrow \bar{T}$ (with zero) on which the Weyl group action on T extends. Since normal torus embeddings have to do with rational polyhedral cones, this yields a discrete geometrical classification of normal connected regular monoids with zero. Renner establishes this classification by first proving a powerful extension theorem: For such monoids M, a homomorphism on G, extending to $\bar{T}$, extends to M.

For the most part we have included all proofs (in many cases simpler than the original), thereby making the book quite appropriate for reading by graduate students. There are a few exceptions. For example, the recent results of the author on conjugacy classes are stated and explained without proofs. However, enough examples are given to give the reader a good understanding. The same is done with a part of Renner's classification theorem.

NOTATION

Throughout this book, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$, $\mathbb{Q}^+$ will denote the sets of all integers, all positive integers, all reals, all positive reals, all rationals, all positive rationals, respectively. If X,Y are sets then $X\backslash Y = \{x \in X \mid x \notin Y\}$. If $Y \subseteq X$, then $X\backslash Y$ will also be denoted by $\sim X$. We let $|X|$ denote the cardinality of X.

K will denote an algebraically closed field, which will remain fixed throughout this book. We let $K^* = K\backslash\{0\}$. If $x_1,\dots,x_n$ are indeterminates, then $K[x_1,\dots,x_n]$ will denote the commutative polynomial algebra in $x_1,\dots,x_n$. If V is a vector space over K, then End(V) will denote the algebra of all linear transformations from V into V, GL(V) its group of units. We let $\mathcal{M}_n(K)$ denote the algebra of all $n \times n$ matrices over K, $K^n = K \times \dots \times K$. If $A \in \mathcal{M}_n(K)$, then A^t, $\rho(A)$, det A will denote the <u>transpose</u> of A, <u>rank</u> of A and <u>determinant</u> of A, respectively. We further let

$$GL(n,K) = \{A \in \mathcal{M}_n(K) \mid \det A \neq 0\}$$
$$SL(n,K) = \{A \in \mathcal{M}_n(K) \mid \det A = 1\}$$
$$\mathcal{T}_n(K) = \{A \in \mathcal{M}_n(K) \mid A \text{ is upper triangular}\}$$
$$\mathcal{D}_n(K) = \{A \in \mathcal{M}_n(K) \mid A \text{ is diagonal}\}$$
$$\mathcal{T}_n{}^*(K) = \mathcal{T}_n(K) \cap GL(n,K)$$
$$\mathcal{D}_n{}^*(K) = \mathcal{D}_n(K) \cap GL(n,K)$$

If $A = (a_{ij}) \in \mathcal{M}_n(K)$, $B \in \mathcal{M}_p(K)$, then $A \otimes B = (a_{ij}B) \in \mathcal{M}_{np}(K)$, $A \oplus B =$

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$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{M}_{n+p}(K)$.

Let $(P, \leq)$ be a partially ordered set. A subset Γ of P is a chain if for all $\alpha, \beta \in \Gamma$ either $\alpha \leq \beta$ or $\beta \leq \alpha$. If Γ is a finite chain, then the length of Γ is defined to be $|\Gamma| - 1$. If $\alpha, \beta \in P$, then α covers β if $\alpha > \beta$ and there is no $\gamma \in P$ with $\alpha > \gamma > \beta$. Let $\alpha, \beta \in P$. If α, β have a greatest lower bound, then this element is denoted by $\alpha \wedge \beta$ and is called the meet of α,β. If α,β have a least upper bound, then this element is denoted by $\alpha \vee \beta$ and is called the join of α,β. If $\alpha \wedge \beta$ exists for all $\alpha, \beta \in P$, then P is a $\wedge$–semilattice. If $\alpha \vee \beta$ exists for all $\alpha, \beta \in P$, then P is a $\vee$–semilattice. If P is both a $\wedge$–semilattice and a $\vee$–semilattice, then it is a lattice. A lattice P is complete if every subset has a least upper bound and a greatest lower bound in P. A lattice P with a maximum element 1 and a minimum element 0 is complemented if for all $\alpha \in P$ there exists $\alpha' \in P$ such that $\alpha \vee \alpha' = 1$, $\alpha \wedge \alpha' = 0$. A lattice P is relatively complemented if for all $\alpha, \beta \in P$ with $\alpha < \beta$, the interval $[\alpha,\beta] = \{\gamma \in P \mid \alpha \leq \gamma \leq \beta\}$ is complemented. A lattice, isomorphic to the lattice of all subsets of a set is called a Boolean lattice.

$\mathcal{B}(T), \mathcal{C}(T)$	Definitions 4.21, 9.9
$\beta(\Lambda), \beta^-(\Lambda), \xi(B), \xi^-(B)$	Definition 9.9
$\mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}, \mid$	Definition 1.1
$\mathcal{P}(B), \mathcal{S}(B) = \mathcal{S}(\Lambda)$	Definition 10.14
$\mathcal{U}$	Definitions 1.5, 12.19, Chapter 14
$\mathcal{N}(E)$	Definition 12.6
$\tilde{\mathcal{U}}$	Definition 10.22
$\hat{\mathcal{U}}$	Definition 14.4
$\hat{E}, E_G, E_{\hat{\mathcal{U}}}$	Chapters 13, 14
$\mathcal{X}(G), \mathcal{X}(M)$	Definition 4.17, Chapter 8
$\mathcal{Y}_\Gamma, \mathcal{Z}_\Gamma$	Definition 11.11
$\phi, \sigma_\alpha, G_\alpha, U_\alpha, T_\alpha (\alpha \in \phi)$	Definitions 4.43, 4.46

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w(width)	Definition 6.26
ht(height), $\mathscr{U}_i$	Definition 6.21
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N(Normalizer)	Chapters 4, 6
C^r, C^ℓ(Right,left centralizer)	Chapter 6