1 ABSTRACT SEMIGROUPS

As usual a set S with an associative operation is called a semigroup. If $\emptyset \neq X \subseteq S$, then $< X >$ will denote the subsemigroup of S generated by X and $E(X) = \{e \in X \mid e^2 = e\}$ the set of idempotents in X. If $e, f \in E(S)$, then $e \geq f$ if $ef = fe = f$. An equivalence relation σ on S is a congruence if for all $a,b,c \in S$, $a \sigma b$ implies $ac \sigma bc$, $ca \sigma cb$. If S' is a semigroup, then a map $\phi: S \to S'$ is a homomorphism if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$. The corresponding congruence is called the kernel of ϕ. A bijection $*: S \to S$ is an involution if $(ab)^* = b^*a^*$, $(a^*)^* = a$ for all $a,b \in S$. A subsemigroup of S which is a group is called a subgroup. S is strongly π–regular ($s\pi r$) if for each $a \in S$, there exists $i \in \mathbb{Z}^+$ such that a^i lies in a subgroup of S. See [1], [19], [49]. If $a,b \in S$, then b is an inverse of a if $aba = a$, $bab = b$. An element $a \in S$ is regular if $axa = a$ for some $x \in S$, i.e. a has an inverse in S. S is regular if each element of S is regular. $\mathcal{M}_n(K)$ is a regular semigroup, and by the Fitting decomposition it is also an $s\pi r$–semigroup. A semigroup with an identity element is called a monoid. If S is a semigroup then $S^1 = S$ if S is a monoid, $S^1 = S \cup \{1\}$ with obvious multiplication if S is not a monoid. Let M be a monoid. An invertible element of M is called a unit. Let G denote the group of units of M. Then M is unit regular if for each $a \in M$, there exists $x \in G$ such that $a = axa$. Equivalently $M = E(M)G$. If M is unit regular, then any submonoid of M containing G is also unit regular.

Definition 1.1. Let S be a semigroup, $a,b \in S$. Then

(i) $a \mathscr{R} b$ if $ax = b$, $by = a$ for some $x,y \in S^1$.

(ii) $a \mathscr{L} b$ if $xa = b$, $yb = a$ for some $x,y \in S^1$.

(iii) $\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$, $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$.

(iv) $a|b$ (a divides b) if $xay = b$ for some $x,y \in S^1$.

(v) $a \mathscr{J} b$ if $a|b|a$; $J_a = \{x \in S | a \mathscr{J} x\}$.

(vi) $J_a \geq J_b$ if $a|b$.

Remark 1.2. For $S = \mathscr{M}_n(K)$, $\mathscr{L}$, $\mathscr{R}$ are row equivalence and column equivalence, respectively. If $a,b \in S$, then $J_a \geq J_b$ if and only if $\rho(a) \geq \rho(b)$.

Remark 1.3. Let S be a semigroup. Then

(i) $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$ are equivalence relations called Green's relations. See [11], [24], [33] for details.

(ii) If $a \in S$, then a lies in a subgroup of S if and only if $a \mathscr{H} e$ for some $e \in E(S)$. In such a case, the $\mathscr{H}$–class of a is the group of units of eSe.

(iii) If S' is an sπr–subsemigroup of S, $a \in S'$, $e \in E(S)$ and if $a \mathscr{H} e$ in S, then $e \in E(S')$ and $a \mathscr{H} e$ in S'.

(iv) Let $a,b,c \in S$. Then $a \mathscr{R} b$ implies $ca \mathscr{R} cb$ and $a \mathscr{L} b$ implies $ac \mathscr{L} bc$.

(v) Let $a \in S$, $e \in E(S)$, $a \mathscr{R} e$, H the $\mathscr{H}$–class of e. Then Ha is the $\mathscr{H}$–class of a.

(vi) Let $e,f \in E(S)$. Then $e \mathscr{R} f$ if and only if $ef = f$, $fe = e$. Similarly $e \mathscr{L} f$ if and only if $ef = e$, $fe = f$.

(vii) Let $a \in S$ be regular. Then $a = axa$ for some $x \in S$. So $e = ax$, $f = xa \in E(S)$, $e \mathscr{R} a \mathscr{L} f$. Thus $a \in S$ is regular if and only if $a \mathscr{R} e$ for some $e \in E(S)$ if and only if $a \mathscr{L} f$ for some $f \in E(S)$.

(viii) Let D be a $\mathcal{D}$–class of S. Then an element of D is regular if and only if each element of D is regular. Let $a \in D$ be regular, x an inverse of a. Then $a \,\mathcal{R}\, ax \,\mathcal{L}\, x$. Hence $x \in D$.

The following well–known result is derived from Green [24], Miller and Clifford [48] and Munn [49].

Theorem 1.4. Let S be an $s\pi r$–semigroup, $a,b,c \in S$. Then

(i) $a \,\mathcal{J}\, ab$ implies $a \,\mathcal{R}\, ab$; $a \,\mathcal{J}\, ba$ implies $a \,\mathcal{L}\, ba$.

(ii) $ab \,\mathcal{J}\, b \,\mathcal{J}\, bc$ implies $b \,\mathcal{J}\, abc$.

(iii) If $e \in E(S)$, J, H the $\mathcal{J}$–class, $\mathcal{H}$–class of e, respectively, then $J \cap eSe = H$.

(iv) $\mathcal{J} = \mathcal{D}$ on S.

(v) $a \,\mathcal{J}\, a^2$ implies that the $\mathcal{H}$–class of a is a group.

(vi) $a \,\mathcal{J}\, ab \,\mathcal{J}\, b$ if and only if $a \,\mathcal{L}\, e \,\mathcal{R}\, b$ for some $e \in E(S)$; $a \,\mathcal{J}\, ba \,\mathcal{J}\, b$ if and only if $a \,\mathcal{R}\, e \,\mathcal{L}\, b$ for some $e \in E(S)$.

(vii) Any regular subsemigroup of S is an $s\pi r$–semigroup.

Proof. (i) Suppose $a \,\mathcal{J}\, ab$. Then $xaby = a$ for some $x,y \in S^1$. Then $x^i a(by)^i = a$ for all $i \in \mathbb{Z}^+$. There exists $j \in \mathbb{Z}^+$ such that $(by)^j \,\mathcal{H}\, e$ for some $e \in E(S)$. Then $a = ae \in a(by)^j S \subseteq abS$. Hence $a \,\mathcal{R}\, ab$.

(ii) By (i), $ab \,\mathcal{L}\, b$. So $abc \,\mathcal{L}\, bc \,\mathcal{J}\, b$.

(iii) If $a \in eSe \cap J$, then by (i), $e \,\mathcal{R}\, ea = a = ae \,\mathcal{L}\, e$. So $a \,\mathcal{H}\, e$.

(iv) Let $a, b \in S$ such that $a \,\mathcal{J}\, b$. Then there exist $x,y \in S^1$ such that $xay = b$. So $a \,\mathcal{J}\, xa \,\mathcal{J}\, xay = b$. By (i), $a \,\mathcal{L}\, xa \,\mathcal{R}\, b$. Hence $a \,\mathcal{D}\, b$.

(v) Let H denote the $\mathcal{H}$–class of a. By (i), $a^2 \,\mathcal{H}\, a$. So $a^2x = a$ for some $x \in S^1$. Then $a^{i+1}x^i = a$ for all $i \in \mathbb{Z}^+$. So $a^i \,\mathcal{R}\, a$ for all $i \in \mathbb{Z}^+$. By (i), $a^i \in H$ for all $i \in \mathbb{Z}^+$. There exists $j \in \mathbb{Z}^+$, $e \in E(S)$ such that $a^j \,\mathcal{H}\, e$. Then $e \in H$ and H is a group.

(vi) Suppose $a \mathcal{J} ab \mathcal{J} b$. Then by (i), $a \mathcal{R} ab \mathcal{L} b$. There exist $x, y \in S^1$ such that $abx = a$, $yab = b$. So $ya = yabx = bx$. Then $aya = a$, $bxb = b$. So $ya \in E(S)$, $a \mathcal{L} ya = bx \mathcal{R} b$. Conversely assume that there exists $e \in E(S)$ such that $a \mathcal{L} e \mathcal{R} b$. So $xa = by = e$ for some $x, y \in S$. Hence $ab | xaby = e | a | ab$. Thus $a \mathcal{J} ab$.

(vii) Let $a \in S'$. There exists $i \in \mathbb{Z}^+$, $e \in E(S)$ such that $b = a^i \mathcal{H} e$ in S. There exists $x \in S'$ such that $b^2xb^2 = b^2$. Then $bxb = e$. So $e \in E(S')$ and $b \mathcal{H} e$ in S'.

Definition 1.5. Let S be an $s\pi r$–semigroup. A $\mathcal{J}$–class J of S is regular if $E(J) \neq \emptyset$. Equivalently some (hence every) element of J is regular. Let $\mathcal{U} = \mathcal{U}(S)$ denote the partially ordered set of all regular $\mathcal{J}$–classes of S. If $J \in \mathcal{U}(S)$, then let $J^0 = J \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } a, b, ab \in J \\ 0 & \text{otherwise} \end{cases}$$

Let S be a semigroup, $\emptyset \neq I \subseteq S$. Then I is a right ideal of S if $IS \subseteq I$; I is a left ideal of S if $SI \subseteq I$; I is an ideal of S if $S^1IS^1 \subseteq I$. The minimum ideal of S, if it exists, is called the kernel of S.

Definition 1.6. (i) A completely simple semigroup S is an $s\pi r$–semigroup with no ideals other than S.

(ii) A completely 0–simple semigroup S is an $s\pi r$–semigroup with 0, having no ideals other than $\{0\}$ and S, and having a non–zero idempotent.

Remark 1.7. (i) This is not the standard definition of completely simple or completely 0–simple semigroups. However this definition is equivalent to the standard one by Munn [49].

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(ii) Let S be an $s\pi r$–semigroup, $J \in \mathcal{U}(S)$. If $a,b \in J$, then there exist $x,s,t \in S^1$ such that $sat = b$, $axa = a$. Then $b = (sax)a(xat) \in JaJ$. Thus J^0 is a completely 0–simple semigroup.

(iii) Let S be an $s\pi r$–semigroup, $J \in \mathcal{U}(S)$. If $E(J)^2 \subseteq J$, then by Theorem 1.4 (ii), $J^2 = J$ and hence J is completely simple.

(iv) A completely simple semigroup has only one $\mathcal{J}$–class while a completely 0–simple semigroup has two $\mathcal{J}$–classes.

Definition 1.8. Let G be a group, Γ, Λ non–empty sets.

(i) Let $P: \Lambda \times \Gamma \to G$ be any map. Let $S = \Gamma \times G \times \Lambda$ with $(i,g,j)(k,h,l) = (i,gP(j,k)h,l)$. Then S is a completely simple semigroup called a Rees matrix semigroup without zero over G (and sandwich map P).

(ii) Let $P: \Lambda \times \Gamma \to G \cup \{0\}$ be any map such that for all $i \in \Gamma$, there exists $j \in \Lambda$ such that $P(j,i) \neq 0$, for all $j \in \Lambda$ there exists $i \in \Gamma$ such that $P(j,i) \neq 0$. Let $S = (\Gamma \times G \times \Lambda) \cup \{0\}$ with

$$(i,g,j)(k,h,l) = \begin{cases} (i,gP(j,k)h,l) & \text{if } P(j,k) \neq 0 \\ 0 & \text{if } P(j,k) = 0 \end{cases}$$

Then S is a completely 0–simple semigroup, called a regular Rees matrix semigroup with zero over G (and sandwich map P).

The following result is due to D. Rees (see [11] or [33]).

Theorem 1.9. (i) Any completely simple semigroup is isomorphic to a Rees matrix semigroup without zero over a group.

(ii) Any completely 0–simple semigroup is isomorphic to a regular Rees matrix semigroup with zero over a group.

Proof. We prove (ii), since (i) follows from it. Let S be a completely 0–simple semigroup. Then $\mathscr{U}(S) = \{J,0\}$ where $J = S\backslash\{0\}$. Let $e \in E(J)$, H,R,L the $\mathscr{H}$–class, $\mathscr{R}$–class, $\mathscr{L}$–class of e, respectively. Let $\Gamma = L/\mathscr{R} = L/\mathscr{H}$, $\Lambda = R/\mathscr{L} = R/\mathscr{H}$. For $\lambda \in \Lambda$, choose $r_\lambda \in \lambda$, for $\gamma \in \Gamma$ choose $l_\gamma \in \gamma$. Let $\lambda \in \Lambda, \gamma \in \Gamma$. If $r_\lambda l_\gamma \in J$, then by Theorem 1.4 (i), $r_\lambda l_\gamma \in H$. Thus we have a map $P: \Lambda \times \Gamma \to H \cup \{0\}$ given by $P(\lambda,\gamma) = r_\lambda l_\gamma$. Let $\lambda \in \Lambda$. Then since r_λ is regular, there exists $f \in E(J)$ such that $r_\lambda \mathscr{L} f$. Since $\mathscr{J} = \mathscr{D}$, there exists $\gamma \in \Gamma$ such that $l_\gamma \mathscr{R} f$. By Theorem 1.4 (vi), $r_\lambda l_\gamma \neq 0$. Similarly for each $\gamma \in \Gamma$, there exists $\lambda \in \Lambda$ such that $r_\lambda l_\gamma \neq 0$. Let $S' = (\Gamma \times H \times \Lambda) \cup \{0\}$ be the Rees matrix semigroup with sandwich map P. Define $\psi: S' \to S$ as $\psi(0) = 0$, $\psi(\gamma,h,\lambda) = l_\gamma h r_\lambda$. Since $ehe = h$ for $h \in H$, we see by Theorem 1.4 that $l_\gamma \mathscr{R}\, l_\gamma h r_\lambda \mathscr{L}\, r_\lambda$. Let $h, h' \in H$, $\gamma \in \Gamma$, $\lambda \in \Lambda$ such that $l_\gamma h r_\lambda = l_\gamma h' r_\lambda$. There exist $y,z \in S$ such that $r_\lambda z = e = y l_\gamma$. It follows that $h = ehe = eh'e = h'$. Thus ψ is injective. That ψ is a homomorphism is immediate. So we need to show that ψ is surjective. Let $a \in J$. There exist $\gamma \in \Gamma$, $\lambda \in \Lambda$ such that $l_\gamma \mathscr{R}\, a\, \mathscr{L}\, r_\lambda$. There exist $y,z \in S$ such that $r_\lambda z = e = y l_\lambda$. Then $ya \,\mathscr{R}\, y l_\gamma = e$, $az \,\mathscr{L}\, r_\lambda z = e$. By Theorem 1.4, $ya \,\mathscr{R}\, yaz \,\mathscr{L}\, az$. Hence $h = yaz \in H$. Now $l_\gamma \mathscr{L}\, e\, \mathscr{R}\, y$ and so $f = l_\gamma y \in E(J)$, $f \mathscr{R}\, l_\gamma \mathscr{R}\, a$. So $l_\gamma ya = a$. Similarly $azr_\gamma = a$. Thus $l_\gamma h r_\lambda = a$. This proves the theorem.

Definition 1.10. Let S be a semigroup, $S = \bigcup_{\alpha \in \Omega} S_\alpha$ a partition of S into subsemigroups. Then S is a semilattice (union) of $S_\alpha (\alpha \in \Omega)$ if for all $\alpha,\gamma \in \Omega$ there exists $\delta \in \Omega$ such that $S_\alpha S_\gamma \cup S_\gamma S_\alpha \subseteq S_\delta$.

Definition 1.11. A semigroup S is completely regular if it is a union of its subgroups.

The following result is due to Clifford [10].

Theorem 1.12. A semigroup S is completely regular if and only if it is a semilattice of completely simple semigroups.

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Definition 1.13. A semigroup S is archimedean if for all $a,b \in S$, $a|b^i$ for some $i \in \mathbb{Z}^+$.

The following result is due to Tamura and Kimura [114].

Theorem 1.14. Any commutative semigroup is a semilattice of archimedean semigroups.

The following result is due to the author [62]. The proof given here is due to Tamura [112].

Theorem 1.15. A semigroup S is a semilattice of archimedean semigroups if and only if for all $a,b \in S$, $a|b$ implies $a^2|b^i$ for some $i \in \mathbb{Z}^+$.

Proof. The necessity of the condition being obvious, assume that for all $a,b \in S$, $a|b$ implies $a^2|b^i$ for some $i \in \mathbb{Z}^+$. Then for all $a,b \in S$, $j \in \mathbb{Z}^+$, there exists $i \in \mathbb{Z}^+$ such that $a^j|b^i$. Define a relation η on S as follows: $a \,\eta\, b$ if $a|b^i$, $b|a^j$ for some $i,j \in \mathbb{Z}^+$. By the above, η is an equivalence relation on S. Let $a,b \in S$. Then $aba|(ab)^2$. So $a^2b|(aba)^2|(ab)^i$ for some $i \in \mathbb{Z}^+$. Continuing, we see that for all $j \in \mathbb{Z}^+$, there exists $k \in \mathbb{Z}^+$ such that $a^jb|(ab)^k$. Now let $a,b \in S$ such that $a \,\eta\, b$. Let $c \in S$. There exists $i \in \mathbb{Z}^+$ such that $a \mid b^i$. Then $xay = b^i$ for some $x,y \in S^1$. So $cxa|cb^i|(cb)^j$ for some $j \in \mathbb{Z}^+$. Hence $ac|(cxa)^2|(cb)^k$ for some $k \in \mathbb{Z}^+$. So $ac|(bc)^{k+1}$. Similarly $bc|(ac)^l$ for some $l \in \mathbb{Z}^+$. Thus $ac \,\eta\, bc$. Similarly $ca \,\eta\, cb$. Hence η is a congruence. Clearly $a \,\eta\, a^2$ for all $a \in S$. Let $a,b \in S$. Then $ab|(ba)^2$, $ba|(ab)^2$. Hence $ab \,\eta\, ba$. It follows that S is a semilattice of its η-classes. Let T be a η-class, $a,b \in T$. Then there exist $x,y \in S$ such that $xay = b^i$. Then $bxayb = b^{i+3}$ and $bx \,\eta\, xay \,\eta\, b \,\eta\, yxay \,\eta\, yb$. So $bx,yb \in T$ and $a|b^{i+3}$ in T. Thus T is an archimedean semigroup. This proves the theorem.

Let S be a semigroup, I an ideal of S. The Rees factor semigroup

$S/I = (S\backslash I) \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } ab \in S\backslash I \\ 0 & \text{otherwise} \end{cases}$$

If S/I is a nil semigroup, then S is a nil extension of I.

Corollary 1.16. Let S be an $s\pi r$–semigroup. Then the following conditions are equivalent.

(i) $E(J)^2 \subseteq J$ for all $J \in \mathscr{U}(S)$.

(ii) For all $a \in S$, $e \in E(S)$, $a \mid e$ implies $a^2 \mid e$.

(iii) S is a semilattice of archimedean semigroups.

(iv) S is a semilattice of nil extensions of completely simple semigroups.

Proof. (i) $\Rightarrow$ (ii). Let $a \in S$, $e \in E(S)$, $a \mid e$. Then $xay = e$ for some $x,y \in S$. So $ayex, yexa \in E(J_e)$. Thus $(yexa)(ayex) \in J_e$ and $a^2 \mid e$.

(ii) $\Rightarrow$ (iii). Let $a,b \in S$ such that $a \mid b$. Then $b^i \mathscr{H} e$ for some $e \in E(S)$, $i \in \mathbb{Z}^+$. So $a \mid e$. Hence $a^2 \mid e \mid b^i$.

(iii) $\Rightarrow$ (iv). Let S_α be an archimedean component of S. Let $a \in S_\alpha$. There exists $e \in E(S)$, $n \in \mathbb{Z}^+$ such that $a^n \mathscr{H} e$ in S. So there exists $x \in S$ such that $a^n x = xa^n = e$, $ex = xe = x$, $ea^n = a^n e = a^n$. It follows that $e,x \in S_\alpha$. Hence S_α is an $s\pi r$–archimedean semigroup. It is obvious that an $s\pi r$–archimedean semigroup is a nil extension of a completely simple semigroup.

(iv) $\Rightarrow$ (i). Let $e,f \in E(S)$, $e \mathscr{J} f$. Then e,f lie in the same archimedean component. Therefore $e \mathscr{J} ef$.

Corollary 1.17. Let S be an $s\pi r$–semigroup which is a semilattice of archimedean semigroups, S' an $s\pi r$–subsemigroup of S. Then S' is a semilattice of archimedean semigroups.

Proof. Let $J \in \mathscr{U}(S')$, $e,f \in E(J)$. Now $(efe)^i \mathscr{H} h$ for some $i \in \mathbb{Z}^+$, $h \in E(S')$. Then $e \geq h$, $e \mathscr{J} h$ in S. So $e = h$ by Theorem 1.4 (i). Hence $ef|e$ and $ef \in J$.

Definition 1.18. Let S, S′ be semigroups, $\phi: S \to S'$ a homomorphism. Then ϕ is idempotent separating if ϕ is 1 − 1 on E(S). A congruence π on S is idempotent separating if for all $e,f \in E(S)$, $e \pi f$ implies $e = f$.

The following result is due to Lallement [40].

Proposition 1.19. Let S, S′ be regular semigroups, $\phi: S \to S'$ a surjective homomorphism which is idempotent separating. Then

(i) $\phi(E(S)) = E(S')$

(ii) If $e,f \in E(S)$, then $\phi(e) \mathscr{R} \phi(f)$ implies $e \mathscr{R} f$; $\phi(e) \mathscr{L} \phi(f)$ implies $e \mathscr{L} f$; $\phi(e) \geq \phi(f)$ implies $e \geq f$.

(iii) If $a,b \in S$, then $\phi(a) = \phi(b)$ implies $a \mathscr{H} b$; $\phi(a) \mathscr{J} \phi(b)$ implies $a \mathscr{J} b$.

Proof. (i) Let $e' \in E(S')$. There exists $a \in S$ such that $\phi(a) = e'$. There exists $x \in S$ such that $a^2xa^2 = a^2$, $xa^2x = x$. Then $e = axa \in E(S)$, $\phi(e) = e'$.

(ii) Let $e,f \in E(S)$, $e' = \phi(e)$, $f' = \phi(f)$. Suppose $e'f' = f'$. Then there exists $x \in S$ such that $(ef)^2x(ef)^2 = (ef)^2$, $x(ef)^2x = x$. Let $f_1 = efx \in E(S)$. Then $\phi(f_1) = f' = \phi(f)$. So $f_1 = f$ and $ef = f$. Similarly $f'e' = f'$ implies $fe = f$.

(iii) Let $a,b \in S$, $\phi(a) = \phi(b)$. There exist $x,y \in S$ such that $axa = a$, $byb = b$. Let $e = ax$, $f = by \in E(S)$. Then $a \mathscr{R} e$, $f \mathscr{R} b$. So $\phi(e) \mathscr{R} \phi(a) = \phi(b) \mathscr{R} \phi(f)$. By (ii), $e \mathscr{R} f$. So $a \mathscr{R} b$. Similarly $a \mathscr{L} b$. Hence $a \mathscr{H} b$. The second statement is now immediate.

Definition 1.20. Let S be a regular semigroup. The congruence μ on S defined by: $a\,\mu\,b$ if and only if $xay\ \mathscr{H}\ xby$ for all $x,y \in S^1$ is called the fundamental congruence on S. S is said to be fundamental if μ is the equality on S.

Remark 1.21. Let S be a regular semigroup. Then

(i) By Proposition 1.19, μ is the largest idempotent separating congruence on S and S/μ is fundamental.

(ii) If $e \in E(S)$, then $\mu|eSe$ is the fundamental congruence on eSe. Let $a \in S$. Then $a\,\mu\,e$ if and only if $a\ \mathscr{H}\ e$ and $af = fa$ for all $f \in E(eSe)$. See Hall [31; Theorem 5].

(iii) If $S = \mathscr{M}_n(K)$, then μ is given by: $a\,\mu\,b$ if and only if $a = \alpha b$ for some $\alpha \in K^*$.

Definition 1.22. A semigroup S is an inverse semigroup if each $a \in S$ has a unique inverse, denoted by a^{-1}.

Remark 1.23. (i) A semigroup S is an inverse semigroup if and only if S is regular and $ef = fe$ for all $e,f \in E(S)$. In such a case $a \to a^{-1}$ is an involution of S. See [11], [33], [61].

(ii) Any commutative idempotent semigroup (called a semilattice) is an inverse semigroup.

(iii) If X is a set, then the semigroup $\mathscr{I}(X)$ of all partial one to one transformations on X is an inverse semigroup, called the symmetric inverse semigroup on X.

(iv) Let E be a semilattice and let T_E denote the subsemigroup of $\mathscr{I}(E)$ consisting of all isomorphisms $\alpha: eE \cong fE$ where $e,f \in E$. T_E is called the Munn semigroup of E.