Chapter 1. Preliminaries

In this chapter we will review some basic facts without proofs and give some of the basic notation that will be used throughout the book. For further results in commutative algebra we refer the reader to the excellent textbooks of Matsumura **[47]**, **[48]** and Nagata **[50]**. For material such as local cohomologies and canonical modules, we recommend Herzog and Kunz **[37]**.

Throughout this chapter R is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and with residue field $k = R/\mathfrak{m}$. We always denote the Krull dimension of R by d. All modules considered here will be finitely generated and unitary.

A. CM modules.

Let M be an R-module. Recall that a sequence $\{x_1, x_2, \dots, x_n\}$ of elements in $\mathfrak{m}$ is a regular sequence on M if x_{i+1} is a non zero divisor on $M/(x_1, x_2, \dots, x_i)M$ for any i $(0 \leq i < n)$. The depth of M is the maximum length of regular sequences on M.

In this book we shall be concerned exclusively with Cohen-Macaulay modules, which are defined as follows:

(1.1) DEFINITION. An R-module M is called a **maximal Cohen-Macaulay module** or simply a **Cohen-Macaulay** (abbr. **CM**) **module** if the depth of M is equal to d. The ring R is a CM ring if R is a CM module over R.

The reader may recall several equivalent definitions of CM modules.

(1.2) PROPOSITION. (Grothendieck **[33]** or Herzog-Kunz **[37]**) *The following conditions are equivalent for an R-module M:*

(1.2.0) *M is a CM module over R;*

(1.2.1) $\mathrm{Ext}^i_R(k, M) = 0 \quad (i < d)$;

(1.2.2) $\mathrm{H}^i_{\mathfrak{m}}(M) = 0 \quad (i \neq d)$,

where $\mathrm{H}^i_{\mathfrak{m}}$ denotes the i-th local cohomology functor with support on $\{\mathfrak{m}\}$.

In practice the conditions (1.2.1) and (1.2.2) will be very useful because of their homological nature. The next two propositions, for example, are proved by using them.

(1.3) PROPOSITION. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of* R*-modules. Then the following hold:*
(1.3.1) *If* L *and* N *are CM, then so is* M.
(1.3.2) *If* M *and* N *are CM, then so is* L.

Warning and Exercise: It is not necessarily true that if L and M are CM, then so is N. Give a counter-example to this.

(1.4) PROPOSITION. *Let* R *be a CM local ring and let*

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0$$

be an exact sequence of R*-modules where each* F_i *is finitely generated free. If* $n \geq d$, *then* M *is a CM module.*

The following facts are rather well known and will be useful later.

(1.5) PROPOSITION.
(1.5.1) *If* R *is a regular local ring, then any CM module over* R *is a free module.*
(1.5.2) *If* R *is a reduced local ring of dimension* 1, *then an* R*-module* M *is CM only when it is torsion free, that is, when the natural homomorphism* $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ *is a monomorphism.*
(1.5.3) *If* R *is a normal local domain of dimension* 2, *then an* R*-module* M *is CM only when it is reflexive, that is, when the natural homomorphism* $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ *is an isomorphism.*
(1.5.4) *If* R *is a normal local domain of dimension* ≥ 3, *then any CM modules over* R *are reflexive. It is, however, not necessarily true that reflexive modules are CM.*

B. Multiplicities.

We now summarize some of the basic results from the theory of multiplicities. For an R-module M, it is known that the length of $M/\mathfrak{m}^n M$ is a polynomial in n if n is large enough, and the polynomial is of the form

$$(e(M)/d!)n^d + (\text{terms of degree less than } d),$$

where $e(M)$ is the **multiplicity** of M. It is true that $e(M)$ is always a nonnegative integer and that $e(M) = 0$ if and only if the Krull dimension $\dim(M)$ is less than d. Multiplicities have the property of additivity in the following sense.

(1.6) PROPOSITION. (Nagata [**50**, Chapter 3])
(1.6.1) *If* $0 \to L \to M \to N \to 0$ *is an exact sequence of R-modules, then the equality* $e(M) = e(L) + e(N)$ *holds.*
(1.6.2) *If R is an integral domain, then we have* $e(M) = e(R)\cdot\mathrm{rank}(M)$ *for any R-module M.*

(1.7) PROPOSITION. (Nagata [**50**, Chapter 3]) *Let M be a CM module over a local ring R. Then, for a system of parameters $\{x_1, x_2, \ldots, x_d\}$ for R, we have the inequality*

$$e(M) \leq length(M/(x_1, x_2, \ldots, x_d)M) \leq n^d \cdot e(M),$$

where n is the least integer with the property $\mathfrak{m}^n \subset (x_1, x_2, \ldots, x_d)R$.

C. Noetherian normalization.

Later in this book we will have occasion to encounter the case where the local ring R is an algebra over another local ring T. In such a case we have the invariance of the CM property.

(1.8) PROPOSITION. (Grothendieck [**33**, Cor.5.7]) *Suppose R is a finite T-algebra where T is also a local ring with the same dimension d. Then an R-module M is CM over R if and only if it is CM over T.*

In fact this proposition can be proved by using (1.2). Under the same assumption as in (1.8), the natural ring homomorphism $T \to R$ is called a **Noetherian normalization** of R when T is regular. It is a classical result that Noetherian normalizations of R exist if R is a complete local domain (Matsumura [**47**, (28.P)]). Combining (1.8) with (1.5.1) we have:

(1.9) PROPOSITION. *Let $T \to R$ be a Noetherian normalization of R. Then an R-module M is CM over R only when it is free regarded as a T-module.*

D. Local duality and canonical modules.

There is a powerful theorem called local duality for CM modules over CM rings. Before stating it we recall the definition of canonical modules.

(1.10) DEFINITION. A module K_R over a CM ring R is a **canonical module** of R if the following two conditions are satisfied:

(1.10.1) K_R is a CM module, and

(1.10.2) $\mathrm{Ext}^i_R(k, K_R) \simeq \begin{cases} 0 & (i \neq d), \\ k & (i = d). \end{cases}$

It is known that this definition is equivalent to the following single equality (Herzog-Kunz [**37**]).

(1.10.3) $\mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(R), E_R(k)) \simeq \widehat{K}_R$,

where $E_R(k)$ is the injective envelope of an R-module k, and $\widehat{K}_R$ denotes the completion of K_R with respect to the $\mathfrak{m}$-adic topology.

In general a canonical module need not exist, but if it does, it is unique up to isomorphism. Fortunately we know that most rings possess canonical modules.

(1.11) PROPOSITION. (Herzog-Kunz [**37**]) *Suppose a CM local ring R has a Noetherian normalization $T \to R$. Then the R-module $K_R = \mathrm{Hom}_T(R, T)$ is a canonical module of R.*

We are now ready to state the theorem of local duality.

(1.12) PROPOSITION. (Grothendieck [**33**, Theorem 6.3]) *Suppose a CM local ring R has the canonical module K_R. Then, for any R-module M, there are natural isomorphisms*

$$\mathrm{Ext}^i_R(M, K_R)^\wedge \simeq \mathrm{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(M), E_R(k)),$$

for any i.

As an easy consequence of this we obtain the following:

(1.13) COROLLARY. *Let M be a CM module over a CM ring R having the canonical module K_R. Then $\mathrm{Hom}_R(M, K_R)$ is also a CM module and there is an isomorphism $M \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(M, K_R), K_R)$. Moreover each $\mathrm{Ext}^i_R(M, K_R)$ vanishes unless $i = 0$. In particular $\mathrm{Hom}_R(\ \ , K_R)$ is an exact auto-functor on the category of CM modules over R.*

For later use we make the following remark which is also an easy corollary of (1.12).

(1.14) REMARK. Let R be the same as in (1.12) and let $0 \to K_R \to L \to M \to 0$ be an exact sequence of CM R-modules where K_R is the canonical module of R. Then the sequence splits.

Actually this is the consequence of the fact that $\mathrm{Ext}^1_R(M, K_R) = 0$. In other words, *the canonical module is an injective object in the category of CM modules.*

E. Syzygies.

(1.15) DEFINITION. Consider an exact sequence of R-modules;

$$0 \longrightarrow N \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where each F_i is a free R-module. Then N is called an n-th **syzygy** of M. The **reduced** n-th syzygy $\mathrm{syz}^n(M)$ of M is the module obtained from N by avoiding all free direct summands. Thus $\mathrm{syz}^n(M)$ has no free direct summand and $N \simeq \mathrm{syz}^n(M) \oplus F$ for some free module F. Notice that the reduced n-th syzygy of M is uniquely determined by M and n up to isomorphism. Note that, by definition, if M is free, then $\mathrm{syz}^n(M) = 0$ $(n \geq 0)$. Note also that if M has projective dimension p, then $\mathrm{syz}^n(M) = 0$ for any $n \geq p$.

Proposition (1.4) can be put in the following form in terms of syzygies.

(1.16) PROPOSITION. *Let R be a CM local ring of dimension d. Then for any R-module M and for any integer n that is not less than d,* $\mathrm{syz}^n(M)$ *is either a CM module or a null module.*

This provides a possible way of constructing new CM modules by taking syzygies or reduced syzygies.

F. Henselian rings.

(1.17) DEFINITION. A local ring R is a **Henselian ring** if the following condition is satisfied.

(1.17.1) Any commutative R-algebra which is module-finite over R is a direct product of local R-algebras.

Recall that an R-module is called **indecomposable** if it has no nontrivial direct summands. There is a crucial fact concerning indecomposability of modules over a Henselian ring.

(1.18) PROPOSITION. *Let R be a Henselian local ring and let M be an R-module. Then M is indecomposable if and only if the endomorphism ring* $\mathrm{End}_R(M)$ *is a local algebra, that is, sums of nonunits in* $\mathrm{End}_R(M)$ *are nonunits. This assures us that the category of finitely generated R-modules admits the Krull-Schmidt theorem. Namely, any R-module is uniquely a finite direct sum of indecomposable R-modules.*

A famous theorem of Hensel asserts that complete local rings are Henselian rings. More generally, analytic algebras defined below are also Henselian.

(1.19) DEFINITION. Let k be a valued field. Thus there is a mapping v from k to the set of nonnegative real numbers, which satisfies the conditions:

(1.19.1) $v(x) = 0$ if and only if $x = 0$. And $v(xy) = v(x) \cdot v(y)$, $v(x+y) \leq v(x) + v(y)$ for any $x, y \in k$.

Then consider a formal power series f over k in n variables $\{x_1, x_2, \ldots, x_n\}$ with the following condition:

(1.19.2) Write f as $\sum a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ ($a_{i_1 i_2 \dots i_n} \in k$). Then there are positive real numbers $r_1, r_2, \dots, r_n$ and N such that $v(a_{i_1 i_2 \dots i_n}) r_1^{i_1} r_2^{i_2} \dots r_n^{i_n} \leq N$ for all $i_1, i_2, \dots, i_n$.

Call such series a **convergent power series** with respect to the valuation v. We denote by $k\{x_1, x_2, \dots, x_n\}$ the set of all convergent power series, and call it a **convergent power series ring**. Notice that if the valuation is trivial, i.e. $v(x) = 1$ for any $x \neq 0$, then all formal power series are convergent ones, thus $k\{x_1, x_2, \dots, x_n\}$ is the formal power series ring.

An **analytic algebra** over k is defined to be a finite algebra over a convergent power series ring. Any complete local ring containing a field is an analytic algebra with trivial valuation. It is known that a local analytic algebra is a Henselian ring (Nagata [**50**, Chapter 7]). If k is a perfect field, then every local analytic algebra is a homomorphic image of a convergent power series ring over k (Scheja-Storch [**57**]). This, however, is not true unless k is perfect. It is also known by Scheja-Storch [**57**, (8.10)] that all analytic algebras are excellent rings.

(1.20) DEFINITION AND PROPOSITION. (Scheja-Storch [**57**]) Let R be a local analytic algebra over a valued field k. Then a system of parameters $\{x_1, x_2, \dots, x_d\}$ for R is called **separable** if the total quotient ring of R is a separable algebra over the quotient field of the convergent power series ring $k\{x_1, x_2, \dots, x_d\}$. If k is a perfect field, then every reduced analytic algebra over k has a separable system of parameters.

G. Split morphisms.

We end this chapter by making several remarks about split morphisms. Recall first its definition. A homomorphism $f : M \to N$ of R-modules is a **split epimorphism** if it has a right inverse, that is, if there is a morphism g from N to M such that $f \cdot g = 1_N$. Similarly f is a **split monomorphism** if f has a left inverse. Notice that if f is a split epimorphism (resp. a split monomorphism), then N (resp. M) is isomorphic to a direct summand of M (resp. N).

For later use we remark the following:

(1.21) PROPOSITION. *Let R be a Henselian local ring and let $f : M \to L$ and $g : N \to L$ be homomorphisms of R-modules which are not split epimorphisms. Suppose L is an indecomposable R-module. Then the homomorphism $(f, g) : M \oplus N \to L$ is not a split epimorphism.*

This is evident from (1.18). Actually, if (f, g) were a split epimorphism, then there would be $a : L \to M$ and $b : L \to N$ satisfying $f \cdot a + g \cdot b = 1_L$. Since $\mathrm{End}_R(L)$ is local, this would imply that either $f \cdot a$ or $g \cdot b$ is an automorphism on L, which is a contradiction.

The following is also easy to see.

(1.22) PROPOSITION. *Assume that* $f : N \to M$ *is a homomorphism of* R*-modules. Let* $N = \sum_i N_i$ *be a direct decomposition of* N *into indecomposable modules and let* $M_j = M/f(\sum_{i \neq j} N_i)$*. Consider* R*-homomorphisms* $f_j : N_j \to M_j$ *for any* j *which are induced naturally from* f*. If all of the* f_j *are split monomorphisms, then so is* f*.*

Chapter 2. AR sequences and irreducible morphisms

This chapter introduces some of the basic theory of AR sequences, which will play the key role in later part of this book. Auslander and Reiten introduced this notion for their theory of representations of Artinian algebras. They, and several others, developed the theory to much wider classes of categories, including the category of CM modules, see [5] for instance. In what follows, 'AR' always stands for 'Auslander and Reiten'.

In this chapter R is always a Henselian CM local ring with maximal ideal $\mathfrak{m}$ and with residue field k. We always denote the Krull dimension of R by d. It is convenient now to introduce the notation for categories of modules. The category of all finitely generated R-modules and R-homomorphisms will be denoted by $\mathfrak{M}(R)$. The full subcategory of $\mathfrak{M}(R)$ consisting of all CM modules will be denoted by $\mathfrak{C}(R)$. Notice that $M \in \mathfrak{C}(R)$ is indecomposable if and only if $\mathrm{End}_R(M)$ is a local ring, cf. (1.18).

Before giving a precise definition of AR sequences it is necessary to introduce a further notion.

(2.1) DEFINITION. For an indecomposable CM module $M \in \mathfrak{C}(R)$, we define a set of short exact sequences $S(M)$ as follows:

$$S(M) = \{s : 0 \to N_s \to E_s \to M \to 0 \mid s \text{ is a } \textit{nonsplit} \text{ exact sequence in } \mathfrak{C}(R) \text{ with } N_s \text{ indecomposable}\}.$$

In particular, an element of $S(M)$ gives a nontrivial element of $\mathrm{Ext}^1_R(M, N_s)$.

The next is a direct consequence of the definition.

(2.2) LEMMA. *If M is an indecomposable CM module over R which is not free, then $S(M)$ is nonempty.*

PROOF: Let s be a nonsplit exact sequence $0 \to N \to E \to M \to 0$ in $\mathfrak{C}(R)$ which ends in M. Since M is nonfree, there exists at least one such exact sequence. For example, it is enough to take E as a free cover of M. Decompose N into indecomposable modules as $N = \sum_i N_i$ and let E_j be $E/\sum_{i \neq j} N_i$. Consider new exact sequences $s_j : 0 \to N_j \to E_j \to M \to 0$. Since s is nonsplit, one of the sequences s_j is also nonsplit (1.22) and thus it lies in $S(M)$. ∎

(2.3) DEFINITION. Let s and t be two elements of $S(M)$.
(2.3.1) We write $s > t$ if there is an $f \in \mathrm{Hom}_R(N_s, N_t)$ such that $\mathrm{Ext}^1_R(M, f)(s) = t$. In this case we say that s is *bigger* than t or t is *smaller* than s. This is equivalent to the existence of a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_s & \longrightarrow & E_s & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_t & \longrightarrow & E_t & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

(2.3.2) We write $s \sim t$ if f is an isomorphism above in (2.3.1). We often identify s with t when $s \sim t$.

(2.4) LEMMA. *Let M be an indecomposable CM module and let s and t be in $S(M)$. If $s > t$ and $t > s$, then we have $s \sim t$. In particular $S(M)$ is a well-defined partially ordered set.*

PROOF: There are, by definition, R-homomorphisms $f : N_s \to N_t$ and $g : N_t \to N_s$ satisfying $\mathrm{Ext}^1_R(M, f)(s) = t$ and $\mathrm{Ext}^1_R(M, g)(t) = s$. If we denote the composition $g \cdot f$ by h, then we have $\mathrm{Ext}^1_R(M, h)(s) = s$. Since both N_s and N_t are indecomposable, it is enough to show that h is an isomorphism. Thus the lemma follows from the following:

(2.5) LEMMA. *Let s be an element of $S(M)$ and let h be an endomorphism of N_s. If $\mathrm{Ext}^1_R(M, h)(s) = s$, then h is an automorphism of N_s.*

PROOF: Suppose h is not an isomorphism. Then h belongs to the Jacobson radical of $\mathrm{End}_R(N_s)$ and hence some power of h is in $\mathfrak{m}\,\mathrm{End}_R(N_s)$. We may thus assume that h itself is in $\mathfrak{m}\,\mathrm{End}_R(N_s)$. Then, for any integer n, we have $h^n = \sum_i a_{in} g_{in}$ for some $a_{in} \in \mathfrak{m}^n$ and $g_{in} \in \mathrm{End}_R(N_s)$. Therefore $s = \mathrm{Ext}^1_R(M, h^n)(s) = \sum_i a_{in} \mathrm{Ext}^1_R(M, g_{in})(s)$ is in $\mathfrak{m}^n \mathrm{Ext}^1_R(M, N_s)$ for any n. Thus we conclude that $s = 0$ as an element of $\mathrm{Ext}^1_R(M, N_s)$ which is clearly a contradiction, for s is nonsplit. ∎

The partially ordered set $S(M)$ has the following property:

(2.6) LEMMA. *Let M be an indecomposable CM module and let $s : 0 \to N_s \to E_s \xrightarrow{p} M \to 0$ and $t : 0 \to N_t \to E_t \xrightarrow{q} M \to 0$ be in $S(M)$. Then there is an element $u \in S(M)$ such that $s > u$ and $t > u$.*

PROOF: Consider an exact sequence $0 \to N \to E \xrightarrow{\varphi} M \to 0$, where $E = E_s \oplus E_t$ and N is the kernel of the homomorphism $\varphi = (p, q)$. Decompose N into indecomposable modules as $N = \sum_i N_i$ and denote $E_j = E/\sum_{i\neq j} N_i$. Then we know from (1.22) that one of the sequences $u_j : 0 \to N_j \to E_j \to M \to 0$ is in $S(M)$, say $u_1 \in S(M)$. Then, from the definition, it is easy to see that $s > u_1$ and $t > u_1$. ∎

(2.7) COROLLARY. *Let M be an indecomposable CM module. If s is a minimal element in $S(M)$, then it is minimum in $S(M)$.*

We are now ready to define AR sequences.

(2.8) DEFINITION. Let M be an indecomposable CM module over R. A short exact sequence $s : 0 \to N_s \to E_s \to M \to 0$ is an **AR sequence ending in** M if s is the minimum element in $S(M)$. An AR sequence ending in M is, if it exists, uniquely determined by M. In particular the modules N_s and E_s are also unique up to an isomorphism. If s is the AR sequence ending in M, then we denote N_s by $\tau(M)$ and call it the **AR translation** of M.

This definition of AR sequences looks very theoretical. We shall rewrite it for practical use.

(2.9) LEMMA. *Let M be an indecomposable CM module and let $s : 0 \to N_s \to E_s \xrightarrow{p} M \to 0$ be in $S(M)$. Then the following two conditions are equivalent:*
(2.9.1) *s is the AR sequence ending in M;*
(2.9.2) *For any R-homomorphism $q : L \to M$ in $\mathfrak{C}(R)$ which is not a split epimorphism, there is an R-homomorphism $f : L \to E_s$ such that $q = p \cdot f$.*

PROOF: (2.9.2) ⇒ (2.9.1) Let $t : 0 \to N_t \to E_t \xrightarrow{q} M \to 0$ be a sequence in $S(M)$ with $t < s$. We want to show that $s < t$. Since q is not a split epimorphism, we know from (2.9.2) that there is an $f : E_t \to E_s$ satisfying $q = p \cdot f$. Denote by $g : N_t \to N_s$ the restriction of f on N_t. Then it follows that $\mathrm{Ext}^1_R(M, g)(t) = s$, that is, $s < t$.

(2.9.1) ⇒ (2.9.2) Let $q : L \to M$ be a homomorphism which is not a split epimorphism. We construct a new exact sequence

$$u : 0 \longrightarrow Q \longrightarrow E_s \oplus L \xrightarrow{\varphi} M \longrightarrow 0,$$

where φ denotes the homomorphism (p, q) and Q is the kernel of φ. Since both p and q are nonsplit, the sequence u is also nonsplit by (1.21). Denoting by h the restriction of the natural monomorphism $E_s \to E_s \oplus L$ on N_s, we see that h is a homomorphism from N_s into Q with the property $\mathrm{Ext}^1_R(M, h)(s) = u$. Decompose Q into indecomposable modules to write $Q = \sum_i Q_i$, hence $\mathrm{Ext}^1_R(M, Q) = \sum_i \mathrm{Ext}^1_R(M, Q_i)$. Write $u = \sum_i u_i$ along this decomposition. Since $u \neq 0$ in $\mathrm{Ext}^1_R(M, Q)$, one of the u_i is nonsplit. Denote it by t. It can be seen from the definition that t is an element of $S(M)$ and that $s > t$. Then