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Introduction to

higher order categorical logic

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Contents

Preface

This book makes an effort to reconcile two different attempts to come to grips with the foundations of mathematics. One is mathematical logic, which traditionally consists of proof theory, model theory and the theory of recursive functions; the other is category theory. It has been our experience that, when lecturing on the applications of logic to category theory, we met with approval from logicians and with disapproval from categorists, while the opposite was the case when we mentioned applications of category theory to logic. Unfortunately, to show that the logicians' viewpoint is essentially equivalent to the categorists' one, we have to slightly distort both. For example, categorists may be unhappy when we treat categories as special kinds of deductive systems and logicians may be unhappy when we insist that deductive systems need not be freely generated from axioms and rules of inference. The situation becomes even worse when we take the point of view of universal algebra. For example, combinatory logics are for us certain kinds of algebras, which goes against the grain for those logicians who have spent a life-time studying what we call the free such algebra. On the other hand, cartesian closed categories and even toposes are for us also certain kinds of algebras, although not over sets but over graphs, and this goes against the grain of those categorists who like to think of products and the like as being given only up to isomorphism. To make matters worse, universal algebraists may not be happy when we stress the logical or the categorical aspects, and even graph theorists may feel offended because we have had to choose a definition of graph which is by no means standard.

This is not the first book on categorical logic, as there already exists a classical monograph on first order categorical logic by Makkai and Reyes, not to mention a book on toposes written by a categorist (Johnstone) and a book on topoi written by a logician (Goldblatt), both of whom mention the internal language of toposes*. Our point is rather this: logicians have made

* Let us also draw attention to the important recent book by Barr and Wells, which manages to cover an amazing amount of material without explicit use of logical tools, relying on embedding theorems instead.

three attempts to formulate higher order logic, in increasing power: typed λ-calculus, Martin–Löf type theory and the usual (let us say intuitionistic) type theory. Categorists quite independently, though later, have developed cartesian closed categories, locally cartesian closed categories and toposes. We claim here that typed λ-calculi and cartesian closed categories are essentially the same, in the sense that there is an equivalence of categories (even untyped λ-calculi are essentially the same as certain algebras we call C-monoids). All this will be found in Part I. We also claim that intuitionistic type theories and toposes are closely related, in as much as there is a pair of adjoint functors between their respective categories. This is worked out in Part II. The relationship between Martin–Löf type theories and locally cartesian closed categories was established too recently (by Robert Seely) to be treated here. Logicians will find applications of proof theory in Part I, while many possible applications of proof theory in Part II have been replaced by categorical techniques. They will find some mention of model theory in Part I and more in Part II, but with emphasis on a categorical presentation: models are functors. All discussion of recursive functions is relegated to Part III.

We deliberately excluded certain topics from consideration, such as geometric logic and geometric morphisms. There are other topics which we omitted with some regret, because of limitations of time and space. These include the results of Robert Seely already mentioned, Gödel's *Dialectica interpretation* (1958), which greatly influenced much of this book, the relation between Gödel's double negation translation and double negation sheaves noted by Peter Freyd, Joyal's proof of Brouwer's principle that arrows from R to R in the free topos necessarily represent continuous functions (and related results), the proof that N is projective in the free topos and the important work on graphical algebras by Burroni.

Of course, like other authors, we have some axes to grind. Aside from what some people may consider to be undue emphasis on category theory, logic, universal algebra or graph theory, we stress the following views:

We decry overzealous applications of Occam's razor.

We believe that type theory is the proper foundation for mathematics.

We believe that the free topos, constructed linguistically but determined uniquely (up to isomorphism) by its universal property, is an acceptable universe of mathematics for a moderate intuitionist and, therefore, that Platonism, formalism and intuitionism are reconcilable philosophies of mathematics.

This may be the place for discussing very briefly who did what. Many results in categorical logic were in the air and were discovered by a number of people simultaneously. Many results were discussed at the Séminaire Bénabou in Paris and published only in preprint form if at all. (Since we are referring to a number of preprints in our bibliography, we should point out that preliminary versions of portions of this book had also been circulated in preprint form, namely Part I in 1982, Part II in 1983 and Part 0 in 1983.) If we are allowed to say to whom we owe the principal ideas exposed in this monograph, we single out Bill Lawvere, Peter Freyd, André Joyal and Dana Scott, and hope that no one whose name has been omitted will be offended.

Let us also take this opportunity to thank all those who have provided us with some feedback on preliminary versions of Parts 0 and I. Again, hoping not to give offence to others, we single out for special thanks (in alphabetic order) Alan Adamson, Bill Anglin, John Gray, Bill Hatcher, Denis Higgs, Bill Lawvere, Fred Linton, Adam Obtułowicz and Birge Zimmermann-Huysgen. We also thank Peter Johnstone for his astute comments on our seminar presentation of Part II. Of course, we take full responsibility for all errors and oversights that still remain.

Finally let us thank Marcia Rodríguez for her conscientious handling of the bibliography, Pat Ferguson for her excellent and patient typing of successive versions of our manuscript and David Tranah for initiating the whole project.

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Montreal, July, 1984

This reprint differs from the original only in the correction of some typographical errors. July 1987

In this reprinting we have repaired various minor misprints and errata. We especially thank Johan van Bentham, Kosta Došen, and Makoto Tatsuta for their careful reading of the text.

Since this book was first published, there has been a tremendous increase of interest in categorical logic among theoretical computer scientists. Of particular importance has been the development of higher-order (= polymorphic) lambda calculi (see Girard's thesis). In the terminology of Part I of this book, such calculi correspond to the

deductive systems associated with the intuitionistic type theories of Part II (cf. also R.A.G. Seely, *J. Symb. Logic* **52** (1987), pp. 969–989).

The equational treatment of weak natural numbers objects in Part I has been extended to strong natural numbers objects (see J. Lambek, *Springer LNM* **1348** (1988) 221–229).

Montreal, March, 1994.