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Iterative Functional Equations

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PREFACE

The present book contains an outline of the modern theory of iterative functional equations. The expression *functional equations* is here understood in a narrow sense (*equations of finite kind*; see Kuczma [20]). It does not include equations in which infinitesimal operations are performed on the unknown functions. So, e.g., differential equations with transformed argument do not fall under this notion.

Nowadays, mainly owing to various activities of Professor J. Aczél, functional equations have grown to be a large, independent branch of mathematics, with its own methods, rich in results and abounding in applications. In such a large area further subdivisions are indispensable. The main line of division runs between *equations in several variables* in which at least one unknown function depends on fewer variables than the number of independent variables actually occurring in the equation (see Aczél [2], Kuczma [12]) and *equations in a single variable*, which can be written using one independent variable only.

Functional equations containing several variables are dealt with in another Encyclopedia volume written by J. Aczél and J. Dhombres [1]. The reader interested in the history of functional equations can consult Dhombres [4], [2] and also Aczél [3], Aczél–Dhombres [1].

'*Iterative functional equations*' is just another name for functional equations in a single variable (such equations are also referred to as *equations of rank 1*). Thus the subject matter of this book is approximately the same as that of Kuczma [26]. However (because of recent developments in the subject), most results now have a more general form and, of course, a number of results dating from after 1968 are found here. Moreover, we present more examples and applications. On the other hand, we do not consider some topics discussed in Kuczma [26]. In particular these include the problems of continuous iteration (embedding functions in flows and semiflows) and construction of general solutions.

Embedding problems are now at the centre of attention of many researchers and they on their own form a very large topic. Here we were only able to report on some aspects of the theory of continuous iteration; see Section 1.7. Construction of general solutions seems less important for applications; we have decided to concentrate on problems of the uniqueness and existence of solutions, justly considered as fundamental for equations of all kinds.

It would have been impossible to present everything about iterative functional equations. In order to keep this book to a reasonable size we had to make a selection, which is always the case with every book. Such a selection undoubtedly reflects the authors' personal preference. Note that at the end of each chapter there is a special section (Notes) in which some further results are briefly discussed and suitable references are given.

Although finite difference equations belong to the general type discussed in this book (and so some theorems can be applied, in particular, to difference equations), they are not dealt with here separately. Finite differences form an independent branch of mathematics and have their special problems and methods, which are not characteristic of more general iterative functional equations. There exist many books devoted entirely to difference equations and the interested reader is referred, e.g., to Nörlund [1] or Gelfand [1].

On the other hand, there are only a few books in which more general iterative functional equations are treated. We may mention here Ghermănescu [2], Gumowski–Mira [1], Kuczma [26], Maier–Kiesewetter [1], Montel [1], Pelyukh–Šarkovskiĭ [3], Targoński [8]. Every one of these books presents a different approach and contains also results not included in the other, nor in the present work. See also the booklets Neuman [12], Smítal [1].

The list of references at the end of this book is by no means complete. We have listed only those items that are referred to in the text. A fairly complete bibliography on iterative functional equations (except, however, finite differences) up to 1967 can be found in Kuczma [26]. But in the last two decades more publications concerning iterative functional equations have appeared than during the previous two centuries.

The story of this book requires a few words of explanation. Originally the book was written by M. Kuczma; the manuscript was ready by the end of 1976 and early in 1977 was sent to the publisher. However, it was felt that its style did not conform with that of the Encyclopedia series and major revisions were suggested. But in 1978 and 1980 the author suffered two strokes, which have resulted in an almost complete paralysis, and so the task of rewriting (and also updating, for several years had already passed since the original manuscript had been completed) the book had to be commissioned to somebody else. The author's colleagues and co-workers B. Choczewski and R. Ger were chosen for this job. The present Chapters 1, 7 and 11 are the work of R. Ger, the remaining chapters have been written by B. Choczewski. R. Ger also compiled the references. M. Kuczma has read everything and approved the changes. During the work the three authors were in constant contact and so the final text may be regarded as a result of their joint effort.

We want to conclude this preface by saying our thanks to those who have helped us at various stages of the work on this book.

We owe very much to Professor J. Aczél, on whose initiative this book has been written and who, all the time, was always ready to aid us with helpful advice and to ease our way through the meanders of formal matters. Our warmest thanks go to him also for his patience and understanding when the work on this book was getting delayed.

It is a pleasant duty to express here our gratitude to Dr K. Baron, Dr D. Brydak, Dr J. Drewniak, Dr H.-H. Kairies, Dr M. Sablik, Dr A. Smajdor and Dr M. C. Zdun, who have provided us with many useful pieces of information and bibliographical hints. Especially appreciated is the help of Dr K. Baron and Dr M. Sablik.

We entrust our book to the readers with the hope that it will be a help for those who want to apply iterative functional equations, and it will become a key to the enchanted world of iterative functional equations for those more interested in theory

THE AUTHORS

SYMBOLS AND CONVENTIONS

There are no preliminaries necessary to this book. However, the reader is assumed to have a basic knowledge of the undergraduate mathematics at a good university.

Throughout the book some particular sets are denoted by special symbols. $\mathbb{P}$ is the set of all prime numbers. $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of nonnegative integers. $\mathbb{Z}$ is the set of all integers. $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}$ is the set of reals, $\bar{\mathbb{R}} = [-\infty, \infty]$ the set of extended reals, $\mathbb{R}^+ = [0, \infty)$ the set of nonnegative real numbers. $\mathbb{C}$ denotes the set of complex numbers – the complex plane. $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. For $n \in \mathbb{N}$ the symbol $\mathbb{K}^n$ denotes the set of all n-tuples $(\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathbb{K}$, and for $m, n \in \mathbb{N}$ the symbol $\mathbb{K}^{m \times n}$ denotes the set of all $m \times n$ matrices with entries from $\mathbb{K}$.

For $x = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$ the symbol $|x|$ denotes the usual Euclidean norm: $|x| = (\sum_{i=1}^n |\xi_i|^2)^{1/2}$, and for $A \in \mathbb{K}^{m \times n}$ the norm $\|A\|$ is defined as the norm of the corresponding linear operator: $\|A\| = \sup_{|x|=1} |Ax|$. With the addition and multiplication by scalars from $\mathbb{K}$ defined in the usual manner, the sets $\mathbb{K}^n$ and $\mathbb{K}^{m \times n}$ endowed with the respective norms are Banach spaces over $\mathbb{K}$.

If $X \subset \mathbb{K}^n$, and $f: X \to \mathbb{K}^m$, $f = (f_1, \ldots, f_m)$, is a differentiable function, then f' denotes the matrix $(\partial f_i / \partial \xi_j) \in \mathbb{K}^{m \times n}$. Thus f' is a function, $f': X \to \mathbb{K}^{m \times n}$. The symbol $\|\cdot\|$ stands for the norm also in any normed space other than $\mathbb{K}^n$.

The symbol 0 may have three different meanings. It may denote the number zero, or the origin in $\mathbb{K}^n$ (or in a more general linear space), or the function whose value is zero at every point of its domain. In every instance it is clear from the context which meaning we have in mind.

Somewhat similar is Landau's O-symbol (Oh) occurring in asymptotic conditions.

The symbol id denotes the identity function, i.e. the function whose value at every point of its domain is equal to that point. If we want to bring the domain (say, X) into evidence, we write id_X. Thus id_X is the function $\mathrm{id}_X: X \to X$ such that $\mathrm{id}_X(x) = x$ for every $x \in X$.

Open and closed real intervals are written, as usual, with round and square brackets, respectively. If it is not essential whether or not an end belongs to the interval in question, we write simply a vertical line. Thus, for instance, $X = [0, a|$ means that X may be either $[0, a)$ or $[0, a]$. But $|a, \infty|$ always means $|a, \infty)$.

The words increasing, decreasing, monotonic are used in the broader sense. Thus a function f is *increasing* iff (= if and only if) $x_1 < x_2$ implies $f(x_1) \leqq f(x_2)$. If $x_1 < x_2$ implies $f(x_1) < f(x_2)$, f is said to be *strictly increasing*; and similarly for decreasing, monotonic.

The word *vicinity* (of a point ξ in a space endowed with a topology) denotes a neighbourhood of ξ from which the point ξ has been removed. So, if U is a neighbourhood of ξ, then $U \setminus \{\xi\}$ is a vicinity of ξ. Neighbourhood is always understood relatively to the underlying set in which the equation in question is considered. The closure of a set A is denoted by cl A, and its interior by int A.

A sequence with terms a_n will be denoted by $(a_n)_{n \in \mathbb{N}}$ or simply (a_n). If X is a topological space, Y is a metric space, and we are given a sequence of functions $f_n: X \to Y$, $n \in \mathbb{N}$, we say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges to a function $f: X \to Y$ *almost uniformly* (abbreviated to a.u.) in X iff it converges to f uniformly on every compact subset of X.

The abbreviation a.e. denotes, of course, almost everywhere (i.e., with the exception of a set of measure zero).

The symbol $\mathscr{F}(X, Y)$ stands for the family of all mappings from X into Y; $\mathscr{F}(X) := \mathscr{F}(X, X)$ whereas $C^r(X, Y)$, $r \in \mathbb{N}_0$, consists of all r times continuously differentiable mappings from $\mathscr{F}(X, Y)$; $C^r(X) := C^r(X, X)$, $r \in \mathbb{N}$; $C(X, Y) := C^0(X, Y)$, $C(X) := C^0(X)$.

Concerning sums and products, we use the convention that

$$\sum_{i=k}^{m} a_i = 0 \quad \text{and} \quad \prod_{i=k}^{m} a_i = 1 \quad \text{whenever } m < k.$$

The present book is divided into 12 chapters and the introduction (Chapter 0). Each chapter is divided into a number of sections, and most sections are further divided into subsections. Within a section we use separate numerations for formulae, theorems, lemmas, corollaries, remarks and definitions.

The end of each proof is marked by the sign ■.