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Introduction

0.0 Preliminaries

0.0A. Types of equations considered

We will not attempt to define what is a functional equation. Such a definition can be found, e.g., in Aczél [2] or Kuczma [26] (see also Pelyukh–Šarkovskiĭ [3]). However, for our use this definition is too complicated and general. Therefore we shall settle for the indication that (with a few exceptions mentioned below) in the present book we shall study functional equations of the general form

$$F(x, \varphi(x), \varphi(f_1(x)), \ldots, \varphi(f_n(x))) = 0, \qquad (0.0.1)$$

where φ is the unknown function and the remaining functions are considered as given. Only in Chapter 11 do we deal with the case where the functions f_i may depend again on the unknown function φ, whereas in Chapter 12 we touch upon the theory of *functional inequalities*, i.e., the case where instead of the equality sign in (0.0.1) we have the inequality $\leqslant$ or $\geqslant$.

The number n occurring in (1) is referred to as the *order* of the equation. Thus the equation of order 1 becomes

$$F(x, \varphi(x), \varphi(f(x))) = 0. \qquad (0.0.2)$$

Equations of order zero, i.e. those of implicit functions $F(x, \varphi(x)) = 0$, will not be dealt with in the present book.

We consider also equations of infinite order, of the form

$$F(x, \varphi(x), \varphi \circ f(\cdot, x)) = 0, \qquad (0.0.3)$$

where $(f(s, \cdot))_{s \in S}$ is a given family of functions, S being a nonvoid set. As a matter of fact, equation (0.0.3) contains equations (0.0.1) and (0.0.2) as particular cases. Equation (0.0.3) reduces to (0.0.2) when S consists of a single element, to (0.0.1) when S has n elements, and to the equation

$$F(x, \varphi(x), \varphi(f_1(x)), \varphi(f_2(x)), \ldots) = 0$$

when S is countable ($S = \mathbb{N}$). If the cardinality of S is the continuum, (0.0.3) may be regarded as an equation of order continuum.

However, in our book we shall confine ourselves mainly to equations of finite orders. Equations of infinite orders will occur only in Chapter 7.

We shall always consider equation (0.0.1) in the form solved with respect to $\varphi(x)$ –

$$\varphi(x) = h(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))) \tag{0.0.4}$$

– or with respect to $\varphi(f_n(x))$ –

$$\varphi(f_n(x)) = g(x, \varphi(x), \varphi(f_1(x)), \ldots, \varphi(f_n(x))). \tag{0.0.5}$$

It can be shown that (0.0.1) always is equivalent to a system of equations of form (0.0.4) and (0.0.5); see Reghiş–Vuc [1], Balint [1], [2].

The main attention will be paid to the case where x is a real variable and the values of the functions occurring are real. A few sections are devoted to the case of a complex variable, and in some cases we consider equations in more general spaces. There are two main reasons for this fact. Firstly, in the overwhelming majority of applications we meet equations in real or complex variables. Secondly, the uniqueness theorems which form the bulk of the present work require rather rich structures in the domain and in the range of the unknown function φ.

0.0B. Problems of uniqueness

In general, equations of form (0.0.1) have too many solutions. The situation is very well exemplified in the case of the equation

$$\varphi(x+1) = \varphi(x), \tag{0.0.6}$$

which is a particular case of (0.0.2) with $f(x) = x + 1$ and $F(x, y, z) = z - y$. Here every periodic function of period 1 is a solution. Such a function can be prescribed arbitrarily on any segment of length 1, open on one side, and thus the general solution φ of equation (0.0.6) depends on an arbitrary function.

In practice such a situation is disadvantageous. If we are led to a functional equation by a practical problem, we rather need a single well-defined solution which could be taken to represent the solution of the original problem. Therefore we need uniqueness theorems yielding a single solution characterized by a particular additional property.

The conditions which may yield the uniqueness of the solution depend very heavily on the form of the equation considered, on the properties of the given functions, and even on the numerical value of some parameters connected with the equation. For instance, in the case of equation (0.0.6) the situation seems to be hopeless. The general continuous or differentiable solution of equation (0.0.6) still depends on an arbitrary function. Nor does the analyticity condition guarantee the uniqueness of the solution. But if we assume that φ is monotonic then necessarily $\varphi = \text{const}$. This is also the family of solutions of (0.0.6) that approach a finite limit at infinity.

Other equations may behave quite differently. Thus the equation

$$\varphi(2x) = \varphi(x) \tag{0.0.7}$$

has the constant functions as the only continuous solutions $\varphi\colon \mathbb{R} \to \mathbb{R}$. And in the case of the equation (for a fixed $r \in \mathbb{N}$)

$$\varphi(2x) = 2^r \varphi(x) \tag{0.0.8}$$

the constant functions are the only solutions $\varphi\colon \mathbb{R} \to \mathbb{R}$ which are of class C^r, whereas in the class C^{r-1} the solution depends on an arbitrary function. On the other hand, the monotonicity condition does not yield uniqueness of solution for equation (0.0.8).

One of the conclusions resulting from the above examples is that there is no universal uniqueness theorem. We have many uniqueness theorems, which are not comparable; none is more general than the others. The more uniqueness theorems we have at our disposal, the bigger is the chance of finding a suitable one for applications. And it is for this reason that theorems of this kind are our main concern in this book.

Of course, we shall deal also with the existence of solutions, in particular with existence-and-uniqueness theorems. However, occasionally there will appear existence theorems that do not guarantee uniqueness, and vice versa.

0.0C. Fixed points

There is also another important conclusion resulting from the analysis of equations (0.0.6), (0.0.7), (0.0.8). In the case of equation (0.0.6) the requirement of the existence of a limit of φ at infinity cannot be replaced by that of the existence of a limit at any other point. In the case of equations (0.0.7) or (0.0.8) the assumption of the continuity or differentiability on the whole real line may be weakened to the condition of the continuity or differentiability at zero. But again zero cannot be replaced by any other point.

The point $\xi = 0$ for the function $f(x) = 2x$, or $\xi = \infty$ for $f(x) = x + 1$ is distinguished as the fixed point of f, $f(\xi) = \xi$. Thus we are led to the very just observation that the conditions yielding the uniqueness of the solution of an iterative functional equation should always be postulated at, or in a neighbourhood of, the fixed point ξ of the function f. A simple change of variables allows us to shift this fixed point to the origin. Therefore, without loss of generality, we will usually assume that the function f has a fixed point at zero.

0.0D. General solution

We leave out from this book the problem of the construction of the general solution φ (without any further conditions) of (0.0.1) or (0.0.2). In this matter the reader is referred to Kuczma [4], [18], [26, Ch. I], Reghiş–Vuc [1], Balint [1], [2], Pelyukh–Šarkovskiĭ [2], [3]. Sometimes, however, we shall wish to describe the general solution (depending on an arbitrary function) of an equation of form (0.0.2) in a particular class of functions. This is essentially possible in two ways which we explain for the example of the equation

$$\varphi(x+1) = x\varphi(x), \qquad x \in (0, \infty). \tag{0.0.9}$$

The general continuous solution $\varphi\colon (0, \infty) \to \mathbb{R}$ of (0.0.9) can be constructed as follows. Fix arbitrarily an $x_0 \in (0, \infty)$. Then every continuous function $\varphi_0\colon [x_0, x_0 + 1] \to \mathbb{R}$ fulfilling the boundary condition $\varphi_0(x_0 + 1) = x_0\varphi_0(x_0)$ can be uniquely extended onto $(0, \infty)$ to a continuous solution $\varphi\colon (0, \infty) \to \mathbb{R}$ of equation (0.0.9).

The other method consists in expressing the general continuous solution $\varphi\colon (0, \infty) \to \mathbb{R}$ of equation (0.0.9) by the formula

$$\varphi(x) = \hat{\varphi}(x)\Gamma(x),$$

where Γ denotes Euler's gamma function (see Section 10.4), and $\hat{\varphi}\colon (0, \infty) \to \mathbb{R}$ is an arbitrary continuous periodic function of period 1. Here the general continuous solution of (0.0.9) has been expressed with the aid of the general continuous solution of (the simpler) equation (0.0.6).

It depends on the particular situation which of the two methods is preferable. Essentially the first method offers more information about the degree of arbitrariness to which the solution is determined. The second one usually yields more elegant, closed formulae, more convenient to handle. The first approach is represented in Kuczma [26], whereas the second can be found in Pelyukh–Šarkovskiĭ [3]. It may be observed that the first method is not possible in the case of analytic solutions.

0.0E. Solution depending on an arbitrary function

Whereas the intuitive sense of this expression is rather clear, it is difficult to give it a precise meaning (see Kuczma [48]). In the present book we shall follow the definition (in spite of all its deficiencies) given in Choczewski–Kuczma [1] and used also in Kuczma [26, p. 45].

Definition 0.0.1. Let $\mathbf{\Phi}$ be a class of functions and let $\mathbf{\Phi}[A]$ denote the class of the restrictions of the functions in $\mathbf{\Phi}$ to the set A. Let X be a subset of a metric space. We say that a functional equation has in the class $\mathbf{\Phi}[X]$ a *solution depending on an arbitrary function* iff there exists an open set $A \subset X$ such that every function in $\mathbf{\Phi}[A]$ can be extended (not necessarily uniquely) to a solution in $\mathbf{\Phi}[X]$ of the equation in question.

0.1 Special equations

0.1A. Change of variables

This is an important procedure which facilitates solving functional equations. An ingenious change of variables may result in a considerable simplification of the equation considered.

Suppose that we are studying the equation

$$F(x, \varphi(x), \varphi(f(x))) = 0, \tag{0.1.1}$$

for functions $\varphi: X \to Y$, where X and Y are arbitrary sets, and 0 denotes an arbitrary, but fixed, element of Y. Further suppose we are given two bijections $\sigma: X \to S$ and $\tau: Y \to T$ (where S, T are again arbitrary sets), which satisfy the functional equations

$$\sigma(f(x)) = g(\sigma(x)), \tag{0.1.2}$$

$$\tau(F(x, y, z)) = L(x, \tau(y), \tau(z)) \tag{0.1.3}$$

(τ may possibly depend on y). Then equation (0.1.1) goes over to

$$L(\sigma^{-1}(t), \psi(t), \psi(g(t))) = \tau(0), \tag{0.1.4}$$

where $t = \sigma(x)$ and the new unknown $\psi: S \to T$ is connected with φ by the formula $\psi = \tau \circ \varphi \circ \sigma^{-1}$. Equation (0.1.4) is again of form (0.1.1), but now it may happen that the new known functions g, L are considerably simpler than f, F. For instance, we may be able to get equation (0.1.2) and (0.1.3) with linear g and L. Then we speak about a *linearization*.

0.1B. Schröder's, Abel's and Böttcher's equations

The most important equations of linearization are those of Schröder and Abel; see Chapter 8. The equation

$$\sigma(f(x)) = s\sigma(x) \tag{0.1.5}$$

will be referred to as the *Schröder equation* (Schröder [1]). Here s may be a scalar factor, as in the classical Schröder equation, or a linear operator on the range of σ.

The *Abel equation* (Abel [1])

$$\alpha(f(x)) = \alpha(x) + A, \tag{0.1.6}$$

where $A \neq 0$ is a fixed element in the range of α (usually one takes $A = 1$) is more subtle than Schröder's and may often be used when equation (0.1.5) fails.

If the linearization is not possible, one may try to reduce the function f in (0.1.1) to the next simplest form. The reduction to a power function may be realized by means of the *Böttcher equation* (Böttcher [1], [2])

$$\beta(f(x)) = [\beta(x)]^p. \tag{0.1.7}$$

The occurrence in the book of the Abel and Schröder equations and of their applications is indicated in Section 9.0.

0.2 Applications

Functional equations we are dealing with have, first of all, many interesting applications in other branches of mathematics. We indicate in Table 0.1 the sections in which such applications may be found.

Table 0.1

Topic	Section(s)
Characterization of functions	6.1A, 10.1–10.5, 11.8
Dynamical systems	8.2
Ergodic theory	3.7A, 6.2A, 6.9
Functional analysis	5.5BC, 9.5
Functional equations in several variables	9.3, 9.6
Functional inequalities	12.2–12.7, 12.9
Geometry	3.5D, 11.9
Iteration theory	1.7, 8.5, 8.7, 11.1–11.7
Ordinary differential equations	3.7C, 9.4, 9.6
Partial differential equations	4.8B
Probability theory	1.4, 3.7B, 6.1B, 6.9
Stochastic processes	2.1, 2.6

A few examples of applications in behavioural and natural sciences are given in Subsections 0.2A–C.

0.2A. Synthesizing judgements

In group decision-making procedure one meets the problem of synthesizing n quantifiable judgements $x_1, x_2, \ldots, x_n$ into the quantity $F(x_1, x_2, \ldots, x_n)$ (see Aczél–Saaty [1] and also Aczél [4]). The values x_k may be obtained, e.g., by ratio estimation which is made by comparing an object with another according to a given criterion.

The synthesizing function $F\colon P^n \to \mathbb{R}$, where $P := |\bar{a}, \bar{b}|$, $0 < \bar{a} < \bar{b} \leqslant \infty$, should meet several reasonable requirements. Among them are the following:

– separability of individual judgements,

$$F(x_1, x_2, \ldots, x_n) = g(x_1) * g(x_2) * \cdots * g(x_n), \qquad x_k \in P, \quad (0.2.1)$$

where '$*$' is a continuous, associative and cancellative operator on $Q := g(P)$, and Q is assumed to be a proper interval;

– consensus condition (if all judgements are x, then the synthesized judgement should be x too),

$$F(x, x, \ldots, x) = x, \qquad x \in P; \quad (0.2.2)$$

– reciprocity property (if second object is compared to first rather than first to second, both individual and synthesized ratios are replaced by their reciprocals),

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) = \frac{1}{F(x_1, \ldots, x_n)}. \quad (0.2.3)$$

By a result of J. Aczél from 1949, all continuous associative and cancellative operators on an interval Q (which is then necessarily open or half-open) are given by the formula

$$y * z = \varphi^{-1}(\varphi(y) + \varphi(z)), \qquad y, z \in Q,$$

where $\varphi: Q \to \mathbb{R}$ is an arbitrary continuous, strictly monotonic function, and

$$\varphi(Q) = \mathbb{R} \quad \text{or} \quad (-\infty, d| \quad \text{or} \quad |e, \infty), \qquad d \leqslant 0 \leqslant e \tag{0.2.4}$$

(see Aczél [2, pp. 254–67]). Therefore, by (0.2.1),

$$F(x_1, \ldots, x_n) = \varphi^{-1}\left(\sum_{k=1}^{n} \varphi(g(x_k))\right), \qquad x_k \in P. \tag{0.2.5}$$

This, when used in (0.2.2), yields

$$\varphi(x) = n\varphi(g(x)), \qquad x \in P, \tag{0.2.6}$$

so that (0.2.5) becomes the quasiarithmetic mean:

$$F(x_1, \ldots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \varphi(x_k)\right), \qquad x_k \in P. \tag{0.2.7}$$

For the reciprocity property we need P to contain with each element its reciprocal. Thus 1 is in P and 0 is not, and P cannot be half-closed. We claim that P is open. For from (0.2.6) we see that both functions $g: P \to Q$ and $\varphi: Q \to \varphi(Q)$ are homeomorphisms. By this and (0.2.4), both $\varphi(Q)$ and P are either open or half-closed, and we have shown that the latter is not the case.

Combining (0.2.3) and (0.2.7) we get the functional equation for φ:

$$\varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \varphi\left(\frac{1}{x_k}\right)\right) = 1/\varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \varphi(x_k)\right) \tag{0.2.8}$$

for $x_k \in P$ which is now an open (finite or infinite) interval of positive numbers, $1 \in P$. After we execute the transformation

$$t_k = \log x_k, \qquad \psi(t) = \varphi(\exp t),$$

equation (0.2.8) goes over into

$$\psi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \psi(-t_k)\right) = -\psi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \psi(t_k)\right), \qquad t_k \in \log P, \tag{0.2.9}$$

and $\log P$ is an open interval symmetric with respect to the origin.

Obviously, every continuous strictly monotonic odd function ψ satisfies (0.2.9). We are going to show that the general strictly monotonic continuous solution of (0.2.9) (for fixed $n > 1$) on an interval $P' = (-c, c)$, $0 < c \leqslant \infty$, is given by

$$\psi(t) = \alpha\psi_0(t) + \beta, \qquad \alpha \neq 0, \tag{0.2.10}$$

where ψ_0 is an arbitrary continuous strictly monotonic odd mapping from P' into $\mathbb{R}$.

Indeed, if ψ_0 satisfies (0.2.9), then so does every $\alpha\psi_0+\beta$ with arbitrary α and β, $\alpha \neq 0$. Now assume that a ψ satisfies (0.2.9) on P' and rewrite the equation in the form

$$\tilde{\psi}^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\tilde{\psi}(t_k)\right)=\psi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\psi(t_k)\right), \qquad t_k \in P', \tag{0.2.11}$$

where we have put $\tilde{\psi}(t)=\psi(-t)$, so that $\tilde{\psi}^{-1}(x)=-\psi^{-1}(x)$.

It is known by a result of K. Knopp from 1929 (see Hardy–Littlewood–Pólya [1, pp. 66–7]) that (0.2.11) implies

$$\tilde{\psi}(t)=a\psi(t)+b$$

for some constants a and b. We have arrived at the linear iterative functional equation

$$\psi(-t)=a\psi(t)+b, \qquad t \in P' \tag{0.2.12}$$

which is easily solved by a single iteration

$$\psi(t)=\psi(-(-t))=a\psi(-t)+b=a^2\psi(t)+(a+1)b.$$

Since ψ cannot be constant, we get from this $a^2=1$ and $(a+1)b=0$. If $a=1$, then $b=0$ and (0.2.12) shows that ψ is an even function, so it cannot be strictly monotonic. In the case $a=-1$ the function $\psi_0(t)=\psi(t)-b/2$ is odd, by (0.2.12). Thus (0.2.10) holds with $\alpha=1$ and $\beta=b/2$.

Since $\log P$ may serve as P', and equation (0.2.9) goes over into (0.2.8), we get all the quasiarithmetic means (0.2.7) having the reciprocal property (0.2.3):

$$F(x_1,\ldots,x_n)=\exp\left[\psi_0^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\psi_0(\log x_k)\right)\right], \qquad x_k \in P,$$

where ψ_0 is an arbitrary continuous strictly monotonic odd function.

If the variables $x_1,\ldots,x_n$ are measures rather than their ratios, then the reciprocity property is less natural. In this case one may wish to replace (0.2.3) by the condition

$$F(f(x_1),f(x_2),\ldots,f(x_n))=f(F(x_1,x_2,\ldots,x_n)), \tag{0.2.13}$$

where $f\colon P\to P$ is a given continuous strictly monotonic function, and ask for functions (0.2.7) satisfying (0.2.13). Combining (0.2.7) and (0.2.13) we see that the resulting equation for φ is just (0.2.11) with ψ replaced by φ and $\tilde{\psi}$ by $\varphi\circ f$, respectively. By the same argument as previously we get, in place of (0.2.12), the equation

$$\varphi(f(x))=a\varphi(x)+b, \qquad x\in P. \tag{0.2.14}$$

This is Abel's equation (2.6) when $a=1$, whereas if $a\neq 1$ (0.2.14) may be transformed to Schröder's equation

$$\sigma(f(x))=a\sigma(x),$$

where $\sigma(x):=\varphi(x)+b/(1-a)$.

In the case where $f(x)=x^p$, $p\neq -1, 0, 1$, the general continuous strictly monotonic solution to (0.2.14) is described in Aczél–Alsina [1]. The solution depends on an arbitrary function, except for some cases where it does not exist at all.

In the above presentation we have followed the lines of Aczél–Saaty [1]. More information on the multiattribute approach to decision-making may be found, e.g., in the book by T. L. Saaty [1].

Monotonic solutions, however, not necessarily strict, of equations like (0.2.14) are discussed in Section 2.3.

0.2B. Clock-graduation and the concept of chronon

Let A be an observer making observations of a particle B by means of light-signals (see Crum [1]). The light-signal sent from A at a time t returns to A, after having been reflected by B, at a time $g(t)$ by A's clock. If we assume that at the time $t=0$ the particle B leaves A and remains in motion in the time-interval $(0, \xi)$, returning to A at $t=\xi\leqslant\infty$, then the function g fulfils the following conditions:

$$g\colon [0, \xi]\to[0, \xi] \text{ is continuous, } g(0)=0,\ g(\xi)=\xi,\ g(t)>t \text{ in } (0, \xi).$$

Moreover, if B has a continuous positive velocity in the time-interval considered, g will be of class C^1 in $[0, \xi]$ with the positive derivative, whence g is strictly increasing.

Now suppose that there is another observer at B, making observations of the light-signals sent by A. Suppose that the light-signal sent by A at the time t (by A's clock) reaches B at a time $\chi(t)$ (by B's clock). If both clocks are congruent, then the light-signal sent (or reflected) by B at the time t returns to A at the time $\chi(t)$. Hence

$$\chi(\chi(t))=g(t). \tag{0.2.15}$$

Consulting, in particular, Theorem 11.1.1 and Lemma 11.2.2, we see that χ must have similar properties to g. Write $f=g^{-1}$, $\varphi=\chi^{-1}$. Then equation (0.2.15) is equivalent to

$$\varphi(\varphi(t))=f(t). \tag{0.2.16}$$

There is an infinity of continuous solutions $\varphi\colon[0, \xi]\to[0, \xi]$ of equation (0.2.16), and all of them fulfil the condition $f(t)<\varphi(t)<t$ in $(0, \xi)$; see Section 11.2. But if we add the condition

$$|f'(t)-f'(0)|\leqslant Ct^{\delta} \qquad \text{for small } t>0$$

with some positive constants C and δ (it is equivalent to an analogous one for g) then, by Theorem 11.3.2, equation (0.2.15) has a unique solution φ, a continuous self-mapping of $[0, \xi]$, which is of class C^1 in $[0, \xi)$. Thus the knowledge of g allows us to determine χ uniquely. Note that in general the derivative $\chi'(\xi)$ will not exist; see Section 11.4.

A functional equation similar to (0.2.16) appears also in a different context (Targoński [7]). Let X be the set of all possible states of the universe, and let $f\colon X \to X$ describe the state evolution in a unit time-interval. Thus if $x \in X$ is the state of the universe at a time t, then $f(x)$ is the state at the time $t+1$. Now, if $\varphi\colon X \to X$ describes the state evolution in the time-interval $1/N$, then clearly

$$\varphi^N(x) = f(x), \tag{0.2.17}$$

where φ^N denotes the Nth functional iterate of φ. If, for a certain $N \in \mathbb{N}$, equation (0.2.17) has no solution (this actually can happen; see Lemma 11.1.1 and the examples in Sections 11.4 and 11.10), then there is no time-interval of length $1/N$. If N is the largest positive integer for which (0.2.17) has a solution, then $\tau = 1/N$ represents the *chronon*, the smallest, indivisible, nonzero time-interval, the quantum of time, as suggested by the analogous quantum of energy in the quantum theory.

0.2C. Sensation scale and Fechner's law

It is well known that very small changes of the stimulus do not result in changes of the sensory experience. For instance, two sounds differing very little in frequency or amplitude (of sound waves) are felt as equally high or loud. The stimulus can be measured in well-defined physical units (like frequency or amplitude). A measurement of sensory experience, that is a derivation of a sensation scale, is a research topic in psychology (see Luce–Edwards [1]).

Suppose that the stimulus is represented by a number x from an interval $X \subset \mathbb{R}^+$, and let $\alpha(x)$ be a measure of the sensory experience caused by this stimulus. There is a Weber function $w\colon X \to \mathbb{R}^+$ such that a stimulus magnitude $y \geqslant x$ is detected as larger than x if $y \geqslant x + w(x)$, whereas for $x \leqslant y < x + w(x)$ the stimuli x and y are indistinguishable. (The function w is determined statistically from experiments). Write

$$g(x) = x + w(x). \tag{0.2.18}$$

Then the distance $w(x)$ from x to $g(x)$ is the *just noticeable difference* (JND) in the stimulus scale. Fechner's condition (which can be assumed as a definition) says that all JNDs on the sensation scale are equivalent to each other. Analytically this means that

$$\alpha(g(x)) - \alpha(x) = \text{const}.$$

Assuming the constant to be 1, i.e. taking the JND as the unit on the sensation scale, we see that the function α must satisfy the Abel equation

$$\alpha(g(x)) - \alpha(x) = 1. \tag{0.2.19}$$

Let $X = (0, \infty)$ and assume that the function $w\colon X \to X$ is continuous and strictly increasing. Then so are $g\colon X \to X$ given by (0.2.18) and $f := g^{-1}$.