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100 **Analysis and Geometry on Groups**

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PREFACE

Many things could be said about the way this book was written but we shall be brief.

It all started with several lecture courses given by N. Varopoulos at Université Paris VI during the period 1982-87. At the time, Coulhon and Saloff-Coste were post-doctoral students and took notes. An early part of these notes appeared for limited circulation in 1986. It was then decided that, when completed, these notes would be published as a set of graduate "Lecture Notes". The project dragged on for several years; by 1990, through the efforts of Saloff-Coste, enough work had been put into the notes to make them presentable as a real book.

This book is primarily an advanced research monograph. It should be accessible to those graduate students that are prepared to make the personal investment and effort to familiarize themselves with the background material.

N. Varopoulos did very little of the actual writing and did not put any work into the preparation of manuscripts; he is however responsible for most of the new mathematics that is presented here. This mathematical work was done during the 1980s and was built on the following basic material.

Existing semigroup theory, especially Beurling–Deny theory; this is work that was done in the 1950s and 1960s. The work of J. Moser and J. Nash on parabolic equations was also a great inspiration in this context.

The theory of second order subelliptic differential operators and especially the "sum of squares operators". This is work done in the 1960s by L. Hörmander. The Harnack estimates, which are essential for us, were completed by J.-M. Bony a little later. This work has since been further developed by several authors.

Finally, the basic real analysis that we all know and which has its origins in the work of Hardy, Littlewood, Marcinkiewicz, etc. in the 1930s.

In fact, *grosso motto*, the above points and all that goes around them, are the background material to which I referred earlier. Some of it is explained but a beginner will no doubt find our explanations a bit concise. To make all this background really accessible to such a reader would have more than doubled the size of the book, and anyway, none of us was prepared to do it!

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Thanks are due to R. Bruno who had a big influence on the decisive early phase of the research that led to this book.

Finally this book owes a great deal to Cambridge University Press and to its Mathematics Editor, David Tranah. He combines two qualities that very seldom go together: he is very professional in his job and at the same time very relaxed in human relations. It was a pleasure doing business with him.

N. Th. Varopoulos
Paris, April 1, 1992

FOREWORD

This short foreword has two purposes. First of all it will clarify the title of the book which might otherwise be deemed a trifle pretentious. It will also give in a few lines some of the motivating problems and a brief history of the subject over the last ten years that led us to write this book.

Considering groups as geometric objects is an old idea. Indeed, one could say that Lie groups are, after all, more than anything else geometric objects. More recently topologists have made good use of this idea to give geometric proofs of purely algebraic theorems. From our point of view however the starting point for the geometric considerations on groups that we shall be interested in can be traced to a paper by Milnor, from 1968. Milnor showed that the fundamental group of a negatively curved compact manifold is of *exponential growth.*

Let us stop for a minute to discuss this theorem. The fundamental group of any compact manifold is a finitely generated discrete group, G. Our first task is to define a natural distance on such a group. Let G be generated by $g_1, ..., g_k \in G$ and let us decree that the ball of radius n centred at $e \in G$ is exactly the set of group elements of the form $g = g_{i_1}^{\varepsilon_1} \cdots g_{i_p}^{\varepsilon_p}$ $(0 \le p \le n,$ $i_\alpha = 1, 2, ..., k,\ \varepsilon_\alpha = \pm 1)$. Using these balls we can define the distance from e, $d(x, y)$ $(x, y \in G)$ by left translation on G. The number of elements in the above n-ball is denoted by $\gamma(n)$ and referred to as the *growth function* of G (Milnor's theorem asserts that the fundamental groups that he considers satisfy $\gamma(n) \ge e^{cn}$.)

After Milnor's paper the volume growth of a group became a subject of research and in 1981 M. Gromov, in a famous paper, gave an algebraic characterization of the groups that satisfy $\gamma(n) = O(n^A)$ (for some fixed $A \ge 0$).

The next natural geometric question to consider is the isoperimetric inequality on finite subsets $\Omega \subseteq G$ of a (finitely generated discrete) group. Let us define

$$\partial\Omega = \{\omega \in \Omega \mid d(\omega, G \setminus \Omega) \le 1\}$$

and, for some $A \ge 1$, examine the possible validity in G of an inequality of the form:

$$|\Omega|^{\frac{A-1}{A}} \le C|\partial\Omega|; \quad \Omega \subseteq G \qquad \mathrm{Iso}(A):$$

where $|\cdot|$ denotes the cardinality of the set and $C > 0$ is independent of Ω. It is very easy to see that if Iso(A) holds in G then $\gamma(n) \ge C_1 n^A$. (Indeed, Iso(A) implies on $\gamma(t)$ the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \ge C^{-1}\left(\gamma(t)\right)^{\frac{A-1}{A}}$$

which gives the result.) One of the theorems that will be proved in this book says that the converse statement also holds.

Let us be more explicit and, for every $f \in c_0(G)$, define the ℓ_p-norm of the gradient

$$||\nabla f||_p^p = \sum_{d(x,y)\leq 1} | f(x) - f(y) |^p .$$

It is very easy to see that Iso(A) is equivalent to

$$||f||_{\frac{A}{A-1}} \leq C||\nabla f||_1; \quad f \in c_0(G). \qquad \text{Sob}(A):$$

One of the main aims of this book is therefore to prove the following geometric result:

$$\text{Sob}(A) \iff \gamma(n) \geq Cn^A \quad n = 1, 2, \ldots$$

We shall also be concerned with analysis and potential theory on groups. Again one could say that this is a very old subject, indeed $\mathbb{R}^n$ is after all a group. More recently (and more significantly from our point of view) an extensive study of an important class of nilpotent groups has been undertaken by many analysts and basic applications to P.D.E.s and complex analysis were discovered. At more or less the same time probabilists and potential theorists have been examining random walks and potential kernels on groups.

If $\mu \in \mathbb{P}(G)$, for any probability measure on, say the discrete, group G one can consider the random walk on $\{Z_n \in G\}_{n\geq 1}$ defined by

$$\mathbb{P}\,[Z_n = g \,//\, Z_{n-1} = h] = \mu(g^{-1}h).$$

One can then ask all the natural questions. For instance, is this walk transient or recurrent? This is usually referred to as Kesten's problem.

A closely related problem can also be considered. Let M be a Riemannian manifold and assume that M is a normal covering of some compact manifold K. We denote by G the deck transformation group so that $M/G \cong K$. This is the same set-up that we had for Milnor's theorem. What started the research in this book was a theorem (proved in the early 1980s) which stated that the canonical Brownian motion on M is transient (respectively, recurrent) if and only if the random walks of Kesten's problem on G are transient (respectively, recurrent), (for appropriate symmetric measures $\mu \in \mathbb{P}(G)$).

Indeed this theorem showed that there was an intimate connection between discrete combinatorial considerations such as a random walk on a discrete group or a graph, and a continuous set-up where analytic tools are more effective. The way to prove the above theorem is to identify G with a discrete "skeleton" of M and use the "natural discretization" of the problem. The correct framework for that "natural discretization" is the Beurling-Deny theory of Dirichlet spaces. Shortly afterwards the Beurling-Deny theory became fundamental for the whole subject. Indeed, from another point of view, the transience criterion of Beurling and Deny (applied directly to the semigroups of the random walks on G) implies the following result:

Let $\mu \in \mathbb{P}(G)$ be symmetric and assume that it has finite generating support. Then the corresponding random walk on G is transient if and only if the following *a priori* inequality holds:

$$|f(e)| \leq C \cdot ||\nabla f||_2; \quad f \in C_0(G). \qquad \text{(Tran.)}:$$

Observe that the criterion (Tran.) is independent of the particular choice of μ, and only depends on the "geometry of G".

If we compare (Tran.) with (Sob) we see that our geometric theorem implies that all groups for which $\gamma(n) \geq Cn^{2+\varepsilon}$ are transient. This, combined with Gromov's theorem, finally yields that the only groups which are not transient are the finite extensions of $\{0\}$, $\mathbb{Z}$, $\mathbb{Z}^2$.

A critical reader could argue that the above problems are combinatorial and that we have not really been doing analysis but discrete mathematics in disguise. To convince such a reader, let us push the process a stage further and examine the asymptotic behaviour of the convolution powers $\mu^{*n}(e)$ for $\mu \in \mathbb{P}(G)$ as above. The transience or recurrence of the random walk can, after all, be determined from the convergence or divergence of the series $\sum \mu^{*n}(e)$.

For the measures that we have been considering, $\mu^{*n}(e)$ is the $\ell^1 \to \ell^\infty$ convolution operator norm of μ^{*n} and the behaviour of this is, for large n, equivalent to the behaviour of $||T_t||_{1\to\infty}$ where

$$T_t = \exp(-t(\delta - \mu))$$

is the continuous time semigroup attached to the walk. What is central here is the following (functional) analytic result:

$$||T_t||_{1\to\infty} = O(t^{-A/2}) \iff ||f||_{\frac{2A}{A-2}} \leq C||\nabla f||_2; f \in c_0(G)$$

valid for any $A > 2$ and for very general semigroups. This theorem shows the intimate connection that exists between Sobolev inequalities and geometry on the one hand, and the analysis of the asymptotic behaviour of natural semigroups on the other.

Many such natural semigroups can be constructed on Lie groups starting from their infinitesimal generators Δ that are second order left-invariant subelliptic differential operators on the group. To close this circle of ideas, observe finally that the heat diffusion semigroup $e^{-t\Delta}$ on $\mathbb{R}^n$ or more generally on a nilpotent Lie group is the main building block for the real analysis there.

Another consequence of the above theorem is that the behaviour of $\phi(n) = \mu^{*n}(e)$ (as $n \to \infty$) measured on the polynomial scale, is independent of μ.

One feels therefore that it ought to be possible to express $\phi(n)$ $(n \to \infty)$ by some group invariant. It is reasonable to conjecture that

$$\phi(n) \sim [\gamma(\sqrt{n})]^{-1},$$

which is unfortunately only correct if $\gamma(n)$ grows polynomially. Alternatives to this conjecture will be examined later on.

One final aspect of the theory that should be mentioned evolves around the Gaussian, off-diagonal, estimates. These are estimates of the form:

$$\mu^{*n}(g) \leq C^{-1} \exp\left(-C\frac{|g|^2}{n}\right).$$

These estimates are significant when $|g| = d(e,g)$ is large, and they are important if we want to have a complete picture of what is happening. The ideas of E. B. Davies are vital for these estimates. (These ideas have been presented in a recent monograph in the same series.)

The geometric and the analytical aspects of the theory that we have described above are held together not only by conceptual considerations but also by the methods of the proofs. Indeed these proofs are based on both aspects simultaneously and it would be impossible to give them without keeping the two sides in mind all along.

There are many topics closely related to our subject that we have not touched upon at all in this book. For example, Riesz transforms, non-unimodular heat kernels, homogeneous spaces, connections with symmetric spaces, and so on.

To finish on an optimistic note, let us express the hope that these topics, which are now in full development, might one day find their place in another book.