CHAPTER I

INTRODUCTION

I.1 Sobolev inequalities in $\mathbb{R}^n$

One of the aims of this book is to study Sobolev inequalities on Lie groups. It is thus natural to present the situation in the simple case of $\mathbb{R}^n$.

In 1936, S. Sobolev published a paper in which he proved a host of *a priori* inequalities. He showed in particular that

$$||f||_{\frac{np}{n-p}} \leq C||\nabla f||_p, \quad \forall f \in C_0^\infty(\mathbb{R}^n), \tag{1}$$

for all $1 < p < n$. It is easy to see that the inequality

$$||f||_q \leq C||\nabla f||_p, \quad \forall f \in C_0^\infty(\mathbb{R}^n), \tag{2}$$

cannot hold unless $p < n$ and $q = np/(n-p)$. Indeed, if one replaces f by $f_\lambda: x \mapsto f(\lambda x)$, $\lambda > 0$ in (2), one gets

$$\lambda^{-n/q}||f||_q \leq C\lambda^{1-n/p}||\nabla f||_p, \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

This forces the above conditions.

Sobolev's paper does not contain a proof of the case $p = 1$. However, on any Riemannian manifold, the inequality

$$||f||_q \leq C||\nabla f||_1, \quad \forall f \in C_0^\infty(\mathbb{R}^n), \tag{3}$$

is equivalent to the isoperimetric inequality

$$(\mathrm{Vol}_n(\Omega))^{1/q} \leq C\mathrm{Vol}_{n-1}(\partial\Omega),$$

where $\partial\Omega$ is the boundary of a smooth bounded open set Ω. If we let $V(t)$ be the volume of a geodesic ball $B(t)$ of fixed centre and radius t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}\,(B(t)) = \mathrm{Vol}_{n-1}\,(\partial B(t))\,.$$

Hence, setting $\Omega = B(t)$ in the isoperimetric inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) \geq C^{-1}V(t)^{1/q}.$$

This shows that Sobolev's inequality (3) with $q = D/(D-1)$ implies $V(t) \geq ct^D$, which establishes a link between the Sobolev inequality and the volume growth function $V(t)$.

In 1958, E. Gagliardo and L. Nirenberg independently found the following elementary proof of the Sobolev inequality (3) in $\mathbb{R}^2$. For $f \in C_0^\infty(\mathbb{R}^2)$, $(x_0, y_0) \in \mathbb{R}^2$, note that

$$|f(x_0,y_0)| \leq \int_{-\infty}^{+\infty} |\frac{\partial f}{\partial x}(x,y_0)|\,dx, \int_{-\infty}^{+\infty} |\frac{\partial f}{\partial y}(x_0,y)|\,dy.$$

Therefore

$$|f(x_0,y_0)|^2 \le \int_{-\infty}^{+\infty} |\frac{\partial f}{\partial x}(x,y_0)|\,dx \int_{-\infty}^{+\infty} |\frac{\partial f}{\partial y}(x_0,y)|\,dy.$$

Integrating over $\mathbb{R}^2$ with respect to x_0 and y_0, we get

$$||f||_2 \le \left(||\frac{\partial f}{\partial x}||_1 ||\frac{\partial f}{\partial y}||_1\right)^{1/2} \le ||\frac{\partial f}{\partial x}||_1 + ||\frac{\partial f}{\partial y}||_1.$$

The same idea applies in $\mathbb{R}^n$, for $n \ge 3$, where one has to use Hölder's inequality. Finally, the L^p versions (1) all follow from the case $p = 1$. Indeed, applying (3) to f^s, where $f > 0$ and $s > 1$, we get

$$\begin{aligned} ||f||^s_{sn/(n-1)} &\le Cs \int f^{s-1} |\nabla f|\,dx \\ &\le Cs \left(\int f^{(s-1)p'}\right)^{1/p'} \left(\int |\nabla f|^p\right)^{1/p}, \end{aligned}$$

where $\frac{1}{p'} + \frac{1}{p'} = 1$. Choosing $s = p(n-1)/(n-p)$, one finds $(s-1)p' = np/(n-p)$ and thus

$$||f||_{np/(n-p)} \le C\frac{p(n-1)}{n-p} ||\nabla f||_p.$$

When $p = 2$, $||\nabla f||_2^2 = (\Delta f, f) = ||\Delta^{1/2}||_2^2$. The above inequality becomes therefore

$$||f||_{2n/(n-2)} \le C||\Delta^{1/2}||_2, \forall f \in C_0^\infty(\mathbb{R}^n), \tag{4}$$

which is sometimes called a Dirichlet inequality; $(\Delta f, f)$ is the Dirichlet form associated with the heat diffusion semigroup $e^{-t\Delta}$. More generally, for $1 < p < +\infty$, $||\nabla f||_p \simeq ||\Delta^{\frac{1}{2}} f||_p$, $\forall f \in C_0^\infty(\mathbb{R}^n)$. This comes from the fact that the Riesz transforms $\frac{\partial}{\partial x_i}\Delta^{-1/2}$ (the higher dimensional analogues of the Hilbert transform), are bounded in $L^p(\mathbb{R}^n)$ when $1 < p < +\infty$. Together with (1), this yields

$$||f||_{np/(n-p)} \le C||\Delta^{1/2} f||_p, \quad \forall f \in C_0^\infty(\mathbb{R}^n), \tag{5}$$

for $1 < p < n$. However, there are ways to obtain (5) that avoid the use of (4). For instance, it can be shown that the operator $\Delta^{-\alpha/2}$, $0 < \alpha p < n$, has a convolution kernel $k_\alpha(x) = c_\alpha |x|^{-n+\alpha}$. This kernel belongs to the weak L^p space $L^{n/(n-\alpha),\infty}$. Now, convolution with a function in $L^{r,\infty}$ sends L^p into L^q, for $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, provided that $1 < p < +\infty$, $1 < r < +\infty$ and $1 < \frac{1}{p} + \frac{1}{r}$. This yields

$$||f||_q \le C||\Delta^{\alpha/2} f||_p, \quad f \in C_0^\infty(\mathbb{R}^n),$$

for $1 < p < +\infty$, $0 < \alpha p < n$. It is worth emphasizing that the result is false for $p = 1$. In that case, only the weak inequality

$$||f||_{n/(n-\alpha),\infty} \le C||\Delta^{\alpha/2} f||_1, \quad f \in C_0^\infty(\mathbb{R}^n),$$

holds. Finally, we note that for $\alpha p > n$, Hölder continuity estimates hold instead.

I.2 Sobolev inequalities and the heat equation on Lie groups

Let us replace $\mathbb{R}^n$ with a connected unimodular Lie group G. Fix a set $\mathbf{X} = \{X_1, ..., X_k\}$ of left invariant vector fields. Without loss of generality, we may assume that $\mathbf{X}$ generates the Lie algebra of G. If it were not the case, we could always consider the sub-Lie algebra generated by $\mathbf{X}$ and work on the corresponding Lie subgroup of G. In this setting, it makes sense to ask for Sobolev inequalities of the type

$$||f||_q \leq C \sum_{i=1}^{k} ||X_i f||_1, \ \forall f \in C_0^\infty(G). \tag{1}$$

If $\{X_1, ..., X_k\}$ is a *basis* of the Lie algebra, we can endow G with a left invariant Riemannian structure by deciding that this basis is orthonormal. Then $\sum_{i=1}^k ||X_i f||_1 \simeq ||\nabla f||_1$, where $\nabla f = (X_1 f, ..., X_k f)$ is the corresponding Riemannian gradient. Inequality (1) is then equivalent to an isoperimetric inequality and implies that the volume $V(t)$ of the geodesic balls of radius t satisfies $V(t) \geq ct^D$, $\forall t > 0$, where $q = D/(D-1)$.

One of our goals is to show that the reverse implication holds. In fact, even when $\mathbf{X}$ is not a basis but only generates the Lie algebra of G, there is a natural distance associated with $\mathbf{X}$. This distance, sometimes called the control distance, is defined by considering absolutely continuous paths that stay tangent almost everywhere to the fields $X_1, ..., X_k$. Let $V(t)$ be the volume of the balls $B(x, t)$ of radius t for that distance; of course $V(t)$ does not depend on the centre x. We will prove

Theorem *The Sobolev inequality*

$$||f||_{D/(D-1)} \leq C \sum_{i=1}^{k} ||X_i f||_1, \ \forall f \in C_0^\infty(G)$$

is equivalent to the volume growth condition

$$V(t) \geq ct^D, \ \forall t > 0.$$

To prove this theorem, our main tools will be the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) u = 0,$$

where $\Delta = \sum_{i=1}^k X_i^2$, and its fundamental solution h_t i.e. the heat kernel. The left invariance of the equation implies that h_t is a right convolution kernel, and we write $h_t(x,y) = h_t(y^{-1}x)$. In the case of $\mathbb{R}^n$, h_t is the Gauss kernel

$$(2\pi t)^{-n/2} e^{\frac{-|x-y|^2}{4t}}.$$

The property $h_{t+s} = h_t * h_s$ always holds and the heat semigroup H_t can be defined by

$$H_t f = f * h_t.$$

Equivalently, H_t is the Markov semigroup associated with the Dirichlet form $D(f,f) = \sum_{i=1}^k ||X_i f||_2^2$.

The study of the properties of the solutions of the heat equation is of interest in itself, but one of the main themes of this book is the connection between these properties and Sobolev inequalities. The following argument, which borrows an idea of Nash, illustrates this point. Assume that the L^2 Sobolev (or Dirichlet) inequality,

$$||f||^2_{2n/(n-2)} \leq C||\nabla f||_2^2 = (\Delta f, f), \tag{2}$$

holds. Then Hölder's inequality yields

$$||f||_2^{2+4/n} \leq C(\Delta f, f)\, ||f||_1^{4/n}.$$

Setting $v(t) = ||h_t(x,.)||_2^2$, we see that $v'(t) = -2(\Delta h_t, h_t)$ and, since $||h_t||_1 = 1$, the above yields the differential inequality $v(t)^{1+2/D} \leq -c\,v'(t)$. Integrating this, we get

$$v(t) \leq \left(\frac{2}{DC}t\right)^{-D/2}, \ \forall t > 0,$$

and therefore

$$\sup_{x\in G} h_t(x) = h_t(e) = ||h_{t/2}||_2^2 \leq Ct^{-D/2}, \ \forall t > 0. \tag{3}$$

In fact, the hypothesis (2) and the conclusion (3) in the above argument are equivalent properties. This equivalence will be proved in the setting of abstract semigroups.

Theorem *Let* e^{-tA} *be a symmetric submarkovian semigroup acting on* L^2 *of some measure space. For any* $D > 2$, *the following properties are equivalent :*

(i) $||e^{-tA} f||_\infty \leq t^{-D/2}||f||_1$, $\forall f \in L^1$, $\forall t > 0$.

(ii) $||f||_{2D/(D-2)} \leq C'||A^{1/2} f||_2$, $\forall f \in \mathcal{D}(A)$.

This abstract result plays a central rôle in our approach.

Returning to the setting of Lie groups, we want to emphasize that the uniform estimate (3) can be complemented with bounds on $h_t(x)$ which include a Gaussian correction when x goes to infinity.

I.3 Harnack's principle

Let us say that Harnack's principle holds if there exists $C > 0$ such that for all $x \in G$ and $t > 0$, any positive solution u of $\left(\frac{\partial}{\partial t} + \Delta\right) u = 0$ in $]0, 4ts[\times B(x, 2\sqrt{s})$ satisfies

$$\sup_{B(x,\sqrt{s})} u(s, y) \leq C \inf_{B(x,\sqrt{s})} u(2s, y). \tag{1}$$

When it holds, the Harnack principle is a very powerful tool. To illustrate this, let us show how (1) yields bounds on the heat kernel in terms of the volume of balls.

Assume thus that Harnack's principle holds, and apply (1) to $u(t, x) = h_t(x)$, which is a solution in $]0, +\infty[\times G$. We conclude that there exists $C > 0$ such that

$$h_s(e) \leq C \inf_{B(e,\sqrt{s})} h_{2s}(x), \forall s > 0.$$

Integrating this inequality over the ball $B(e, \sqrt{s})$, we obtain

$$V(\sqrt{s}) h_s(e) \leq C \int_{B(e,\sqrt{s})} h_{2s}(x)\, dx \leq C$$

since $||h_{2s}||_1 = 1$. In particular, we see that $h_t(e) \leq Ct^{-D/2}$ as soon as Harnack's principle holds and $V(t) \geq ct^D$.

The drawback is that the Harnack principle does not always hold. Moreover, even if it does, it is not always easy to prove.

In our analytic and geometric study of Lie groups, each question has two different aspects. One corresponds to a local point of view, the other to a global one. The simplest instance of this is the behaviour of the volume growth function $V(t)$ which depends upon whether t tends to zero or to infinity. From a local point of view, the group structure does not play an important rôle, if any. Indeed, we will offer a local study of sums of squares of vector fields on manifolds. This will be based on dilation arguments and a local Harnack principle.

From the global point of view of geometry and analysis at infinity, the group structure enters into play in an essential manner. Indeed, in the setting of Lie groups, our basic result is that the Sobolev inequality

$$||f||_{D/(D-1)} \leq C||\nabla f||_1, \forall f \in C_0^\infty(G),$$

the kernel estimate

$$h_t(e) \leq Ct^{-D/2}, \forall t > 0,$$

and the volume growth condition

$$V(t) \geq ct^D, \forall t > 0$$

are equivalent. This equivalence simply fails to hold if one replaces the group G by – say – a Riemannian manifold with bounded geometry.

I.4 A guide to this book

In Chapter II, we build the semigroup machinery that will enable us to link the Sobolev inequalities with the behaviour of the heat kernel. These functional analytic results are of independent interest.

In Chapter III we describe some basic properties of the sums of squares of vector fields. A given set of vector fields $X_1, \ldots, X_{k+1}$ satisfies the Hörmander condition if the fields $X_1, \ldots, X_{k+1}$ together with their brackets of every order span the tangent space at each point. Under this condition, a genuine distance can be defined by considering the "minimal length" of absolutely continuous paths tangent to the fields $X_1, \ldots, X_{k+1}$. Moreover, the operator $\sum_{i=1}^{k} X_i^2 + X_{k+1}$ is hypoelliptic (Hörmander's theorem) and a local Harnack inequality holds.

Chapter IV focuses on the study of the sublaplacian associated with a Hörmander system of left invariant vector fields on a nilpotent Lie group. Here our analysis is based on Harnack's principle. Indeed, any connected nilpotent Lie group can be covered by another nilpotent Lie group that admits a dilation structure. This dilation structure, together with the local Harnack principle derived in Chapter III, yields the scaled Harnack principle described in Section 3 above. This principle tranfers easily to G. From this, a two-sided Gaussian bound for the heat kernel follows. We also study in detail the volume growth of nilpotent Lie groups. This shows the existence of a local dimension d that governs the behaviour of the volume of small balls, and of a dimension at infinity D that governs the volume of large balls. Finally, heat kernel and volume estimates, together with Chapter II, yield optimal Sobolev inequalities.

Chapter IV also serves as a model for a general study of Hörmander systems of vector fields. In Chapter V, we show how Harnack's principle and a local scaling technique yield satisfactory local results for the heat equation associated with sublaplacians on groups and manifolds.

Chapter VI introduces in the simple setting of discrete groups the main ideas leading to the analytic and geometric study of groups at infinity. In order to stay away from technicalities, the results are not stated in their optimal form. However, they are more than enough to show that the only recurrent finitely generated groups are the finite extensions of $\{0\}, \mathbb{Z}$, and $\mathbb{Z}^2$.

Chapter VII develops the various tools needed to extend and refine the results of Chapter VI. The main result establishes the sharp relationship between volume growth and decay of convolution powers, in the setting of locally compact, compactly generated groups. In the process, we give an analogue of the theory of Chapter II for discrete time semigroups.

Chapter VIII considers unimodular connected Lie groups. Here, the functional analytic tools of Chapter II play an essential part. Together with the local results of Chapter V, they yield, in the case of polynomial volume growth, two-sided Gaussian estimates for the heat kernel, optimal Sobolev inequalities, and Harnack's principle. In the case of exponential volume growth, we prove a sharp result concerning the uniform decay of the heat kernel at infinity. Chapter IX gives up the study of the heat kernel and concentrates on Sobolev inequalities for non-unimodular Lie groups. An inequality of Hardy, ideas from Chapter VII and the splitting of G as the semi-direct product $G \simeq \overline{G} \rtimes R$, where $\overline{G}$ is the kernel of the modular function, are the essential ingredients.

Finally Chapter X contains various geometric applications of the above theory.

This book does not aim at being self-contained. A background in functional analysis, differential geometry and Lie group theory would certainly be helpful to the reader. However, not much is needed to understand our geometric and analytic study of discrete groups in Chapter VI. Also it is *not* necessary to master the theory of one-parameter semigroups of operators to follow Chapter II, and it is certainly not necessary to master the details of Lie group theory to make one's way through Chapters IV, VIII, and IX. The only important result whose proof is not given, but which is nevertheless fundamental (for the local theory) is Hörmander's theorem.

All references, including the bibliography for the background material we use in the text, are to be found in the References and Comments section at the end of each chapter.

CHAPTER II

DIMENSIONAL INEQUALITIES FOR SEMIGROUPS OF OPERATORS ON THE L^p SPACES

II.1 Introduction, notation

Let $T_t = e^{-tA}$ be a symmetric submarkovian semigroup on a measure space (see Section 5 for definitions). The main theme of this chapter is the equivalence between the following two properties, for $n \in]2, +\infty[$:

$$D_n : \quad \exists C \text{ such that } ||f||_{2n/(n-2)} \leq C(Af, f)^{\frac{1}{2}}, \quad \forall f \in \mathcal{D}(A),$$

$$R_n : \quad \exists C \text{ such that } ||T_t f||_\infty \leq Ct^{-n/2}||f||_1, \quad \forall f \in L^1, \forall t > 0.$$

A number n, if there is any, such that D_n or R_n is fulfilled may be called the dimension of the semigroup e^{-tA}: for example, if n is an integer greater than 3, the heat semigroup on $\mathbb{R}^n$, $e^{-t\Delta}$, has dimension n and the Poisson semigroup, $e^{-t\Delta^{\frac{1}{2}}}$, has dimension $2n$. This justifies the title of this chapter. Nevertheless, a semigroup may have no dimension (e.g. the trivial semigroup) or several ones: if the measure space under consideration is discrete, a semigroup of dimension n is also of dimension m, for every $m \leq n$, since the ℓ^p spaces are nested.

The implication "$R_n \Rightarrow D_n$" relies on an abstract analogue of the classical Hardy-Littlewood-Sobolev theory which will be developed in Section 2. The implication "$D_n \Rightarrow R_n$" can be obtained in several ways. We shall present three of them in Section 3. The first one uses the analyticity of the semigroup. The other two are inspired by ideas of Nash and Moser. None of these methods are limited to the framework of symmetric submarkovian semigroups, and this is important for the applications. Nevertheless, this setting is central, and we shall consider it specifically in Section 5. In Section 4, properties D_n and R_n are localized for small and large time.

From now on, (X, ξ) will be a measure space and T_t, $t \geq 0$, a semigroup of operators defined on $L^1 \cap L^\infty$ which, for every $p \in [1, +\infty]$, extends to a semigroup on L^p, of class C^0 if $p \neq +\infty$ (i.e. $T_t f \to f$ in L^p when $t \to 0^+$). Define A_p by $A_p f = \lim_{t \to 0^+} \frac{f - T_t f}{t}$, when this limit exists in L^p; $-A_p$ is the infinitesimal generator of T_t on L^p. Denote by $\mathcal{D}$ a vector space dense in L^p, and dense for the graph norm in the domain of A_p, for every finite p. For instance, it is easy to see that the space

$$\mathcal{D}_0 = \text{Vect}\{\int_0^{+\infty} \phi(t) T_t f \, dt \mid \phi \in C_0^\infty(]0, +\infty[), f \in L^\infty, \xi(\{f \neq 0\}) < +\infty\}$$

always fulfills these conditions. We shall thus omit the index p and set $A = A_p$, $p \in [1, +\infty[$.

A harmonic function with respect to T_t is a function $v: \mathbb{R}^{+*} \times X \to \mathbb{C}$ such that

$$\forall t > 0, \quad v_t(.) = v(t, .) \in L^1 + L^\infty$$

$$\forall t, s > 0, \quad T_t v_s = v_{t+s}.$$

For example, if $f \in L^1 \cap L^\infty$, $v(t,x) = T_t f(x)$ is harmonic with respect to T_t. Given a harmonic function v, denote by v^* the maximal function defined by $v^*(x) = \sup_{t>0} |v(t,(x))|$. For $p \in]0,+\infty[$, we shall say that a function v which is harmonic with respect to T_t belongs to H^p if v^* belongs to L^p. The quasi-norm $||v||_{H^p} = ||v^*||_p$ is a norm when $p \geq 1$.

II.2 Hardy–Littlewood–Sobolev theory

The property R_n is a regularization property which may be generalized for $0 < p < q \leq +\infty$ as follows:

$$R(n,p,q) \quad ||v(t,.)||_q \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})/2} \sup_{s>0} ||v(s,.)||_p, \quad \forall t > 0, \forall v \text{ harmonic.}$$

II.2.1 Proposition *Suppose there exist* $0 < p < q \leq +\infty$ *and* $n > 0$ *such that*

$$||T_t f||_q \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})/2} ||f||_p, \quad \forall t > 0, \forall f \in L^p.$$

Then $R(n,p_1,q)$ *is satisfied for every* $p_1 \leq p$.

Proof Using the hypothesis and Hölder's inequality, one has, putting $\alpha = n(\frac{1}{p}-\frac{1}{q})/2$ and $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{p_1}$,

$$||T_{2t}f||_q \leq Ct^{-\alpha}||T_t f||_p \leq Ct^{-\alpha}||T_t f||_q^\theta ||T_t f||_{p_1}^{1-\theta}.$$

Set

$$K(f,r) = \sup_{t\in[0,r]} \{t^{\alpha/(1-\theta)}||T_t f||_q (\sup_{s>0} ||T_s f||_{p_1})^{-1}\}.$$

One has

$$||T_{2t}f||_q \leq CK^\theta(f,r)t^{-\alpha/(1-\theta)} \sup_{s>0} ||T_s f||_{p_1}, \quad t \in]0,1].$$

Thus

$$K(f,r) \leq CK^\theta(f,r).$$

Therefore $K(f,r) \leq C^{1/(1-\theta)}$ for f such that $\sup_{s>0} ||T_s f||_{p_1} < +\infty$. The property $R(n,p_1,q)$ easily follows.

If T_t is equicontinuous on L^1 and L^∞, $R(n,p,q)$ may be reformulated, for $1 \leq p < q \leq +\infty$, as

$$||T_t||_{p\to q} \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})/2}, \quad \forall t > 0.$$

Thus one has

II.2.2 Proposition *Let* T_t *be a semigroup which is equicontinuous on* L^1 *and* L^∞, *and let* $n > 0$. *Then:*

(i) R_n *is equivalent to each* $R(n,p,q)$, *for* $1 \leq p < q \leq +\infty$.

(ii) R_n *implies* $R(n,p,q)$ *for all* p,q *such that* $0<p<q<+\infty$.

Proof If $0<p<q\le+\infty$, Hölder's inequality

$$||v(t,.)||_q \le ||v(t,.)||_\infty^{1-p/q}||v(t,.)||_p^{p/q},$$

shows that $R(n,p,\infty)$ always implies $R(n,p,q)$. Moreover, the Riesz–Thorin theorem shows that $||T_t||_{\infty\to\infty}\le C_1$ and $||T_t||_{p\to\infty}\le C_2t^{-n/2p}$ imply

$$||T_t||_{r\to\infty}\le Ct^{-n/2r} \quad \text{for } 1\le p\le r\le+\infty.$$

It follows that R_n implies $R(n,p,q)$ for $1\le p<q\le+\infty$ if T_t is equicontinuous on L^∞; Proposition II.2.1 then easily gives (ii). To end the proof of (i), fix $1\le p<q\le+\infty$ and suppose that $R(n,p,q)$ is satisfied. Proposition II.2.1 then shows that $R(n,1,q)$ is satisfied, which means, since T_t is equicontinuous on L^1, that

$$||T_tf||_q\le Ct^{-n/2q'}||f||_1, \quad \forall t>0, \quad \forall f\in L^1,$$

where $\frac{1}{q}+\frac{1}{q'}=1$. The dual semigroup T_t^* thus satisfies

$$||T_t^*f||_\infty\le Ct^{-n/2q'}||f||_{q'}, \quad \forall t>0, \quad \forall f\in L^{q'}.$$

Applying Proposition II.2.1 to T_t^*, one gets

$$||T_t^*f||_\infty\le Ct^{-n/2}||f||_1, \quad \forall t>0, \quad \forall f\in L^1.$$

By duality, one obtains the same estimate for T_t, i.e. T_t satisfies R_n.

We are now going to see that R_n implies mapping properties for the potential operators $G_\zeta f=\Gamma(\zeta/2)A^{-\zeta/2}f$, for $\zeta\in\mathbb{C}$, $\operatorname{Re}\zeta>0$. One may take as a definition of these operators

$$G_\zeta f=\int_0^{+\infty}t^{(\zeta/2)-1}T_tf\,dt$$

when the integral converges at $+\infty$. Note that if v is harmonic and if $G_\zeta v$ exists, $G_\zeta v$ is also a harmonic function.

II.2.3 Proposition *Let* $n>0$ *and* $0<p<+\infty$. *Suppose that* T_t *satisfies* $R(n,p,\infty)$, *and let* $q>0$ *and* $\zeta\in\mathbb{C}$, $\operatorname{Re}\zeta=\gamma>0$, *be such that* $\frac{1}{q}=\frac{1}{p}-\frac{\gamma}{n}$. *Then, for every harmonic function* v *belonging to* H^p, $G_\zeta v$ *exists and*

$$(G_\zeta v)^*(x)\le Cv^*(x)^{p/q}||v^*||_p^{1-(p/q)}.$$

In particular, G_ζ *is bounded from* H^p *to* H^q.

Proof Let us write

$$G_\zeta v(s,x)=\int_0^T t^{(\zeta/2)-1}v(t+s,x)\,dt+\int_T^{+\infty}t^{(\zeta/2)-1}v(t+s,x)\,dt.$$