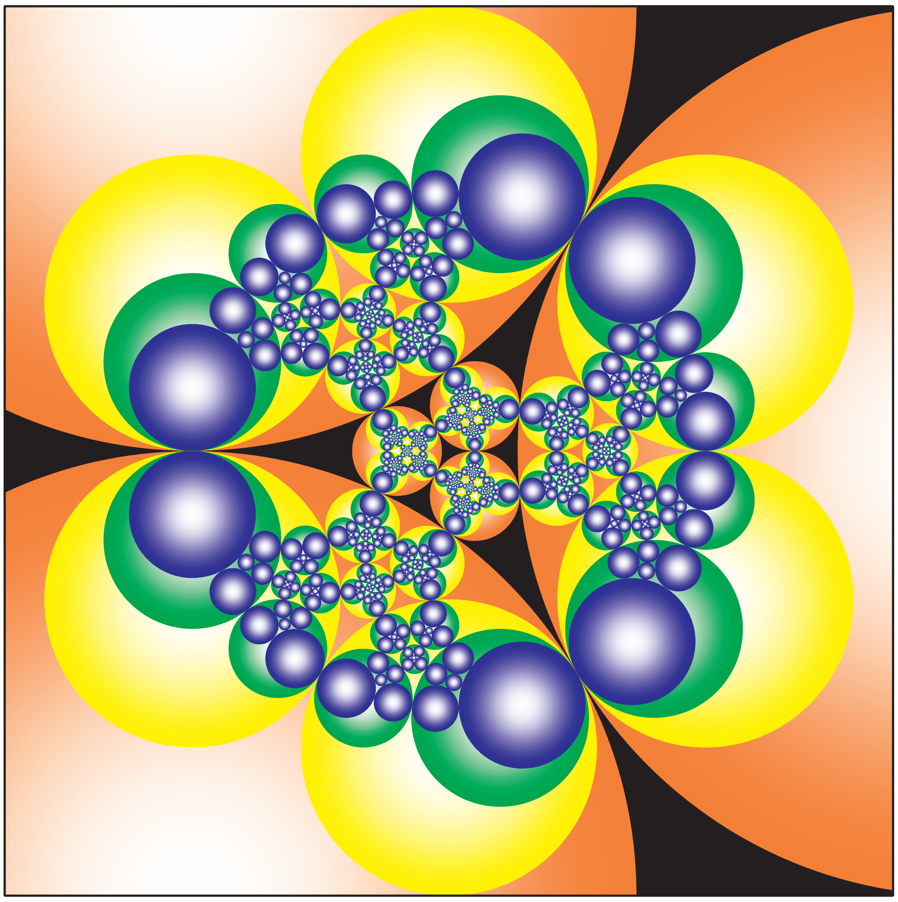
INDRA'S PEARLS

THE VISION OF FELIX KLEIN



The ancient Buddhist dream of Indra's Net

In the heaven of the great god Indra is said to be a vast and shimmering net, finer than a spider's web, stretching to the outermost reaches of space. Strung at each intersection of its diaphanous threads is a reflecting pearl. Since the net is infinite in extent, the pearls are infinite in number. In the glistening surface of each pearl are reflected all the other pearls, even those in the furthest corners of the heavens. In each reflection, again are reflected all the infinitely many other pearls, so that by this process, reflections of reflections continue without end.

Cover picture: A mathematically generated picture foretold in the Buddhist myth of Indra's net? We sometimes call these *Klein Bubbles*. The smallest ones are *sehr klein*.

INDRA'S PEARLS

The Vision of Felix Klein

David Mumford, Caroline Series and David Wright

With cartoons by Larry Gonick

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PREFACE

What kind of a book is this?

This is a book about serious mathematics, but one which we hope will be enjoyed by as wide an audience as possible. It is the story of our computer aided explorations of a family of unusually symmetrical shapes, which arise when two spiral motions of a very special kind are allowed to interact. These shapes display intricate 'fractal' complexity on every scale from very large to very small. Their visualisation forms part of a century-old dream conceived by the great German geometer Felix Klein. Sometimes the interaction of the two spiral motions is quite regular and harmonious, sometimes it is total disorder and sometimes – and this is the most intriguing case – it has layer upon layer of structure teetering on the very brink of chaos.

As we progressed in our explorations, the pictures that our computer programs produced were so striking that we wanted to tell our tale in a manner which could be appreciated beyond the narrow confines of a small circle of specialists. You can get a foretaste of their variety by taking a look at the Road Map on the final page. Mathematicians often use the word 'beautiful' in talking about their proofs and ideas, but in this case our judgment has been confirmed by a number of unbiassed and definitely non-mathematical people. The visual beauty of the pictures is a veneer which covers a core of important and elegant mathematical ideas; it has been our aspiration to convey some of this inner aesthetics as well. There is no religion in our book but we were amazed at how our mathematical constructions echoed the ancient Buddhist metaphor of Indra's net, spontaneously creating reflections within reflections, worlds without end.

Most mathematics is accessible, as it were, only by crawling through a long tunnel in which you laboriously build up your vocabulary and skills as you abstract your understanding of the world. The mathematics behind our pictures, though, turned out not to need too much in the way of preliminaries. So long as you can handle high school algebra with confidence, we hope everything we say is understandable. Indeed given time and patience, you should be able to make programs to create new pictures for yourself. And if not, then browsing through the figures alone should give a sense of our journey. Our dream is that this book will reveal to our readers that mathematics is not alien and remote but just a very human exploration of the patterns of the world, one which thrives on play and surprise and beauty.

And how did we come to write it?

David M.'s story. This book has been over twenty years in the writing. The project began when Benoit Mandelbrot visited Harvard in 1979/80, in the midst of his explorations of complex iteration – the 'fractals' known as Julia sets – and the now famous 'Mandelbrot Set'. He had also looked at some nineteenth century figures produced by infinite repetitions of simple reflections in circles, a prototypical example of which had fascinated Felix Klein. David W. and I pooled our expertise and began to develop these ideas further in the Kleinian context. The computer rapidly began producing pictures like the ones you will find throughout the book.

What to do with the pictures? Two thoughts surfaced: the first was that they were unpublishable in the standard way. There were no theorems, only very suggestive pictures. They furnished convincing evidence for many conjectures and lures to further exploration, but theorems were the coin of the realm and the conventions of that day dictated that journals only publish theorems.[1] The second thought was equally daunting: here was a piece of real mathematics that we could explain to our non-mathematical friends. This dangerous temptation prevailed, but it turned out to be *much, much* more difficult than we imagined.

[1] Since then, the pioneering team of Klaus and Alice Peters have started the journal *Experimental Mathematics*.

We persevered off and on for a decade. One thing held us back: whenever we got together, it was so much more fun to produce more figures than to write what Dave W. named in his computer `TheBook`. I have fond memories of traipsing through sub-zero degree gales to the bunker-like supercomputer in Minneapolis to push our calculations still further. The one loyal believer in the project was our ever-faithful and patient editor, David Tranah. However, things finally took off when Caroline was recruited a bit more than a decade ago. It took a while to learn how to write together, not to mention spanning the gulfs between our three warring operating systems. But our publisher, our families and our friends told us in the end that enough was enough.

You know that 'word problem' you hated the most in elementary school? The one about ditch diggers. Ben digs a ditch in 4 hours, Ned in 5 and Ted in 6. How long do they take to dig it together? The textbook will tell you 1 hour, 37 minutes and 17 seconds. Baloney! We have uncovered incontrovertable evidence that the right answer is (4+5+6)=15 hours. This is a deep principle involving not merely mathematics but sociology, psychology, and economics. We have a remarkable proof of this but even Cambridge University Press's generous margin allowance is too small to contain it.

David W.'s story. This is a book of a thousand beginnings and for a long time apparently no end. For me, though, the first beginning was in 1979 when my friend and fellow grad student at Harvard Mike Stillman told me about a problem that his teacher David Mumford had described to him: Take two very simple transformations of the plane and apply all possible combinations of these transformations to a point in the plane. What does the resulting collection of points look like?

Of course, the thing was not just to think about the shapes but to actually draw them with the computer. Mike knew I was interested in discrete groups, and we shared a common interest in programming. Also, thanks to another friend and grad student Max Benson, I was alerted to a very nice C library for drawing on the classic Tektronix 4014 graphics terminal. The only missing ingredient was happily filled by a curious feature of a Harvard education: I had passed my qualifying exams, and then I had nothing else to do except write my doctoral thesis. I have a very distinct memory of feeling like I had a lot of time on my hands. As time has passed, I have been astonished to discover that that was the last time I felt that way.

Anyway, as a complete lark, I tagged along with David M. while he built a laboratory of computer programs to visualize Kleinian groups. It was a mathematical joy-ride. As it so happened, in the summer of 1980, there was a great opportunity to share the results of these computer explorations with the world at the historic Bowdoin College conference in which Thurston presented his revolutionary results in three-dimensional topology and hyperbolic geometry. We arranged for a Tektronix terminal to be set up in Maine, and together with an acoustically coupled modem at the blazing speed of 300 bits per second displayed several limit sets. The reaction to the limit curves wiggling their way across the screen was very positive, and several mathematicians there also undertook the construction of various computer programs to study different aspects of Kleinian groups.

That left us with the task of writing an explanation of our algorithms and computations. However, at that point it was certainly past time for me to complete my thesis. Around 1981, I had the very good fortune of chatting with a new grad student at Harvard by the name of Curt McMullen who had intimate knowledge of the computer systems at the Thomas J. Watson Research Center of IBM, thanks to summer positions there. After roping Curt in, and at the invitation and encouragement of Benoit Mandelbrot, Curt and David M. made a set of extremely high quality and beautiful black-and-white graphics of limit sets. I would like to express my gratitude for Curt's efforts of that time and his friendship over the years; he has had a deep influence on my own efforts on the project.

Unfortunately, as we moved on to new and separate institutions, with varying computing facilities, it was difficult to maintain the programs and energy to pursue this project. I would like to acknowledge the encouragement I received from many people including my friend Bill Goldman while we were at M.I.T., Peter Tatian and James Russell, who worked with me while they were undergraduates at M.I.T., Al Marden and the staff of the Geometry Center, Charles Matthews, who worked with me at Oklahoma State, and many other mathematicians in the Kleinian groups community. I would also like to thank Jim Cogdell and the Southwestern Bell Foundation for some financial support in the final stages. The serious and final beginning of this book took place when Caroline agreed to contribute her own substantial research work in this area and her expository gifts, and also step into the middle between the first and third authors to at least moderate their tendency to keep programming during our sporadic meetings to find the next cool picture. At last, we actually wrote some text.

We have witnessed a revolution in computing and graphics during the years of this project, and it has been difficult to keep pace. I would also like to thank the community of programmers around the world for creating such wonderful free software such as TEX, Gnu Emacs, X Windows and Linux, without which it would have been impossible to bring this project to its current end.

During the years of this project, the most momentous endings and beginnings of my life have happened, including the loss of my mother Elizabeth, my father William, and my grandmother and family's matriarch Elizabeth, as well as the birth of my daughters Julie and Alexandra. I offer my part in these pictures and text in the hope of new beginnings for those who share our enjoyment of the human mind's beautiful capacity to puzzle through things. Programming these ideas is both vexing and immensely fun. Every little twiddle brings something fascinating to think about. But for now I'll end.

Caroline's story. I first saw some of David M. and David W.'s pictures in the mid-80s, purloined by my colleague David Epstein on one of his periodic visits to the Geometry Center in Minneapolis. I was struck by how pretty they were – they reminded me of the kind of lace work called tatting, which in another lifetime I would have liked to make myself.

I presumed that everyone else understood all about the pictures, and didn't pay too much attention, until a little while later Linda Keen and I were looking round for a new project. I had spent many years working on Fuchsian groups (see Chapter 6), and was wanting something which would lead me in to the Kleinian realm where at that time it was all go, developing Thurston's wonderful new ideas about three-dimensional non-Euclidean geometry (see Chapter 12). By that time, I had somehow got hold of Dave W.'s preprint[1] which described the explorations reported in Chapter 9. I suggested to Linda that it might fit the bill.

The first year was one of frustration, staring at pictures like the ones in Chapter 9 without being able to get any real handle on what was going on. Then one morning one of us woke up with an idea. We tried a few hand calculations and it seemed promising, so we asked Dave W. to draw us a picture of what we called the 'real trace rays'. What came back was a rudimentary version of the last picture in this book – the one we have called 'The end of the rainbow'. For me it was more like 'The beginning of the rainbow', one of the defining moments of my mathematical life. Here we were, having made a total shot in the dark, having no idea what the rays could mean, but knowing they had absolutely no right to be arranged in such a nice way. It was obvious we had stumbled on something important, and from that moment, I was hooked.

For another year we struggled to fit the rays into the one mathematical straight-jacket we could think of, but it just didn't quite work. One day, I ran into Curt McMullen and mentioned to him what we were playing with. 'Real trace', he pondered, 'That's the convex hull boundary'.[2] And with that clue, we were off. What Curt had told us was that to understand the two dimensional pictures we had to look in three-dimensional non-Euclidean space, real Thurston stuff, as you might say. Finally we were able to verify at least most of the two Daves' conjectures theoretically.

When the 19th century mathematician Mary Somerville received a letter inviting her to make a translation, with commentary, of Laplace's great book *Mécanique Célèste*, she was so surprised she almost returned the letter thinking there must have been some mistake.[3] I suppose I wasn't quite so surprised to get a letter from David M. asking me to help them write about their pictures, but it wasn't quite an everyday occurrence either.

[1] *The shape of the boundary of the Teichmüller space of once- punctured tori in Maskit's embedding*, Unpublished preprint.

[2] See Chapter 12 for a bit more explanation and some pictures.

[3] Fortunately, her enlightened husband convinced her otherwise. The book became a best seller and was even published illegally in the US!

Although I may perhaps write another book, I am unlikely ever again to have the chance to work on one which will be so much trouble and so much fun.

And don't think this book is the end of the story. If you flick through you will see cartoons of a rather portly character gluing up pieces of rubber into things like doughnuts. In fact all our present tale revolves about 'one-holed doughnuts with a puncture'. For the last few years, I have been trying to understand what happens when the doughnuts acquire more holes. The main thing I can report is – it's a lot more complicated! But the same wonderful structures, yet more intricate and inviting, are out there waiting to be tamed.

I would like to thank the EPSRC for the generous support of a Senior Research Fellowship, which has recently allowed me to devote much time to both the mathematical and literary aspects of this challenging project.

Guide to the reader

This is a book which can be read on many levels. Like most mathematics books, it builds up in sequence, but the best way to read it may be skipping around, first skimming through to look at the pictures, then dipping in to the text to get the gist and finally a return to understand some of the details. We have tried to make the first part of each chapter relatively simple, giving the essence of the ideas and postponing the technicalities until later. The more technical parts of the discussion have been relegated to the Notes and can be skipped as desired. Material important for later reference is displayed in Boxes.

The first two chapters, on Euclidean symmetries and complex numbers respectively, contain material which may be partially familiar to many readers. We have aimed to present it in a form suited to our viewpoint, at the same time introducing as clearly as possible and with complementary graphics the mathematical terminology which will be used throughout the book. Chapter 3 introduces the basic double spiral maps, called Möbius symmetries, on which all of our later constructions rest. From then on, we build up ever more complicated ways in which a pair of Möbius maps can interact, generating more and more convoluted and intricate fractals, until in Chapters 10 and 11 we actually reach the frontiers of current research. The entire development is summarised in the Road Map on the final page.

Words which have a precise mathematical meaning are in **bold face** the first time they appear. We have not always spelled out the intricacies of the precise mathematical definition, but we have also tried not to say anything which is mathematically incorrect. We have used a small amount of our own terminology, but in so far as possible have stuck to standard usage. Non-professional readers will therefore have to forgive us such terms as quasifuchsian and modular group, while readers with a mathematical training should be able to follow what we mean.

The book is written as a guide to actually coding the algorithms which we have used to generate the figures. A vast set of further explorations is possible for those readers who invest the time to program. This is prime hacking country! Because we hope for a wide variety of readers with many different platforms at their disposal, we have sketched each step in 'pseudo-code', the universal programming pidgin.

Inevitably we have suppressed a good deal of relevant mathematics and anyone wishing to pursue these ideas seriously will doubtless sooner or later have to resort to more technical works. Actually there are no very accessible books about Kleinian group limit sets[1], but there are plenty of texts which discuss the basics of symmetry and complex numbers. Some complex analysis books touch on Möbius maps and there is more in modern books on two-dimensional hyperbolic geometry. In the later part of the book we have cited a rather random collection of recent research papers which have important bearing on our work. These are absolutely not meant to be exhaustive, but should serve to help professional readers find their way round the literature.

[1]For readers with mathematical training the best introduction may still be Lester Ford's 1929 *Automorphic Functions*, Chelsea reprint, 1951.

Finally our Projects need some comment. They can be ignored: we aren't going to grade them or supply answers! Rather, we intend them as 'explorations' to tempt you if you enjoy the material and want to take it further. Some are fairly straightforward extensions or elucidations of material in the text and some involve open-ended questions for which there is no definite answer. A few are definitely research problems. Others again explain details which are needed for full understanding or verification of the more technical points in our story. We have to leave it to the reader to pick and choose which ones suit their taste and mathematical experience.

Acknowledgements

We thank especially our cartoonist Larry Gonick for his uncanny ability to translate a complicated three-dimensional manipulation into an immediately evident cartoon. For historical background we are indebted to the St. Andrews History of Maths web site, tempered with many erudite details and healthy doses of scholarly scepticism from our friends David Fowler and Paddy Patterson. (All remaining errors, are, of course, our own.) Klein's own book *Entwicklung der Mathematik im 19. Jahrhundert*

has also been an important source. We have read the Hua-Yen Sutra in the translation *The Flower Ornament Scripture* by Thomas Cleary, Shambhala Publications, 1993, and quotations are reproduced here with thanks. We should like to thank the Mathematics Departments of Brown, Oklahoma State, Warwick, Harvard and Minnesota for their hospitality. We should like to thank the NSF through its grant to the Geometry Center and EPSRC from their Public Understanding of Science budget for financial support. Finally we should like to thank our publisher David Tranah of Cambridge University Press, without whose constant prodding and encouragement this book would almost certainly never have seen the light of day.

Acknowledgments for the paperback edition

In this paperback edition, we would like sincerely to thank the many people who have read the book and sent corrections and suggestions, especially Yoshiaki Araki, Michael Barnsley, Peter Cromwell, John Guenther, Roger House, George Jackson, Arny Katz, Marius Kempe, Yohei Komori, Bill Margolis, Jon V. Pepper, Stuart Price and Masaaki Wada.

We would like particularly to thank Isabel Seliger for locating bibliographic information regarding one of the commentaries in Chinese Huayan Buddhist literature that explicates the metaphor of Indra's pearls.[1] The text elaborates the imagery of the reflecting pearls in ways that are strikingly close to contemporary mathematics. We are especially struck by this sentence in Tanabe's translation: "Within the boundaries of a single jewel are contained the unbounded repetition and profusion of the images of all the jewels." Could there be a better summary of the mathematics you will find below?

[1] *Calming and Contemplation in the Five Teachings of Huayan* (華嚴五教止觀, Huayan wujiao zhiguan), in Taishō shinshū daizōkyō (Buddhist canon newly compiled under the Taishō reign era [1912–26]), eds. Takakusu Junjirō and Watanabe Kaigyoku (Tokyo: Taishō issai-kyō kankō-kai, 1914–22), vol. 45, no. 1867, pp. 513a28–513b21. The Chinese source text can be accessed online through the Chinese Buddhist Electronic Text Association (CBETA) website at `www.cbeta.org/result/normal/T45/1867_001.htm`. An English translation by George Tanabe appears in *Sources of Chinese Tradition: From Earliest Times to 1600*, vol. 1, ed. W. T. De Bary (New York: Columbia University Press, 1999), p. 473.

INTRODUCTION

I have discovered things so wonderful that I was astounded ... Out of nothing I have created a strange new world.

János Bolyai

With these words the young Hungarian mathematical prodigy János Bolyai, reputedly the best swordsman and dancer in the Austrian Imperial Army, wrote home about his discovery of non-Euclidean geometry in 1823. Bolyai's discovery indeed marked a turning point in history, and as the century progressed mathematics finally freed itself from the lingering sense that it must describe only the patterns in the 'real' world. Some of the doors which these discoveries flung open led directly to new worlds whose full exploration has only become possible with the advent of high speed computing in the last twenty years.

Paralleling the industrial revolution, mathematics grew explosively in the nineteenth century. As yet, there was no real separation between pure and applied mathematics. One of the main themes was the discovery and exploration of the many special functions (sines, cosines, Bessel functions and so on) with which one could describe physical phenomena like waves, heat and electricity. Not only were these functions useful, but viewed as more abstract entities they took on a life of their own, displaying patterns whose study intrigued many people. Much of this had to do with understanding what happened when ordinary 'real' numbers were replaced by 'complex' ones, to be described in Chapter 2.

A second major theme was the study of symmetry. From Mayan friezes to Celtic knotwork, repeating figures making symmetrical patterns are as ancient as civilization itself. The Taj Mahal reflects in its pool, floors are tiled with hexagons. Symmetry abounds in nature: butterfly wings make perfect reflections and we describe the tile pattern as a honeycomb. The ancients already understood the geometry of symmetry: Euclid tells us how to recognise by measurement when two triangles are congruent or 'the same' and the Alhambra displays many mathematically different ways of covering a wall with repeating tiles.

The nineteenth century saw huge extensions of the idea of symmetry and congruence, drawing analogies between the familiar Euclidean world and others like Bolyai's new non-Euclidean universe.[1] Around the middle of the century, the German mathematician and astronomer August Möbius had the idea that things did not have to be the same shape to be identified: they could be compared as long as there was a definite *verwandschaft* or 'relationship' between every part of one figure and every part of the other. One particular new relationship studied by Möbius was inspired by cartography: figures could be considered 'the same' if they only differed by the kind of distortions you have to make to project figures from the round earth to the flat plane. As Möbius pointed out (and as we shall study in Chapter 3), these special relationships, now called Möbius maps, could be manipulated using simple arithmetic with complex numbers. His constructs made beautifully visible the geometry of the complex plane.[2]

[1]Non-Euclidean geometry was actually discovered independently and at more or less the same time by Gauss, Bolyai and Lobachevsky, see Chapter 12.

[2]The complex plane is pictured in Figure 2.1

Towards the end of the century, Felix Klein, one of the great mathematicians of his age and the hero of our book, presented in a famous lecture at Erlangen University a unified conception of geometry which incorporated both Bolyai's brave new world and Möbius' relationships into a wider conception of symmetry than had ever been formulated before. Further work showed that his symmetries could be used to understand many of the special functions which had proved so powerful in unravelling the physical properties of the world (see Chapter 12 for an example). He was led to the discovery of symmetrical patterns in which more and more distortions cause shrinking so rapid that an infinite number of tiles can be fitted into an enclosed finite area, clustering together as they shrink down to infinite depth.

It was a remarkable synthesis, in which ideas from the most diverse areas of mathematics revealed startling connections. Moreover the work had other ramifications which were not to be understood for almost another century. Klein's books (written with his former student Robert Fricke) contain many beautiful illustrations, all laboriously calculated and drafted by hand. These pictures set the highest standard, occasionally still illustrating mathematical articles even today. However many of the objects they imagined were so intricate that Klein could only say:

> The question is ... what will be the position of the limiting points. There is no difficulty in answering these questions by purely logical reasoning; but the imagination seems to fail utterly when we try to form a mental image of the result.[3]

[3]*The mathematical character of space-intuition*, Klein, *Lectures on Mathematics*, 1894, Reprinted by AMS Chelsea, 2000.

The wider ramifications of Klein's ideas did not become apparent until two vital new and intimately linked developments occurred in the 1970's.

The first was the growing power and accessibility of high speed computers and computer graphics. The second was the dawning realization that chaotic phenomena, observed previously in isolated situations (such as theories of planetary motion and some electronic circuits), were ubiquitous, and moreover provided better models for many physical phenomena than the classical special functions. Now one of the hallmarks of chaotic phenomena is that structures which are seen in the large repeat themselves indefinitely on smaller and smaller scales. This is called self-similarity. Many schools of mathematics came together in working out this new vision but, arguably, the computer was the *sine qua non* of the advance, making possible as it did computations on a previously inconceivable scale. For those who knew Klein's theory, the possibility of using modern computer graphics to actually *see* his 'utterly unimaginable' tilings was irresistible.

Our frontispiece is a modern rendering of one of Klein's new symmetrical worlds. In another guise, it becomes the *The Glowing Limit* shown overleaf. Peering within the bubbles, you can see circles within circles, evoking an elusive sense of symmetry alongside the self-similarity characteristic of chaos. Without the right mathematical language, though, it is hard to put one's finger on exactly what this symmetry is. The sizes and positions of the circles in the two pictures are not the same: the precise *verwandschaft* between them results from the distortion allowed by a Möbius map.

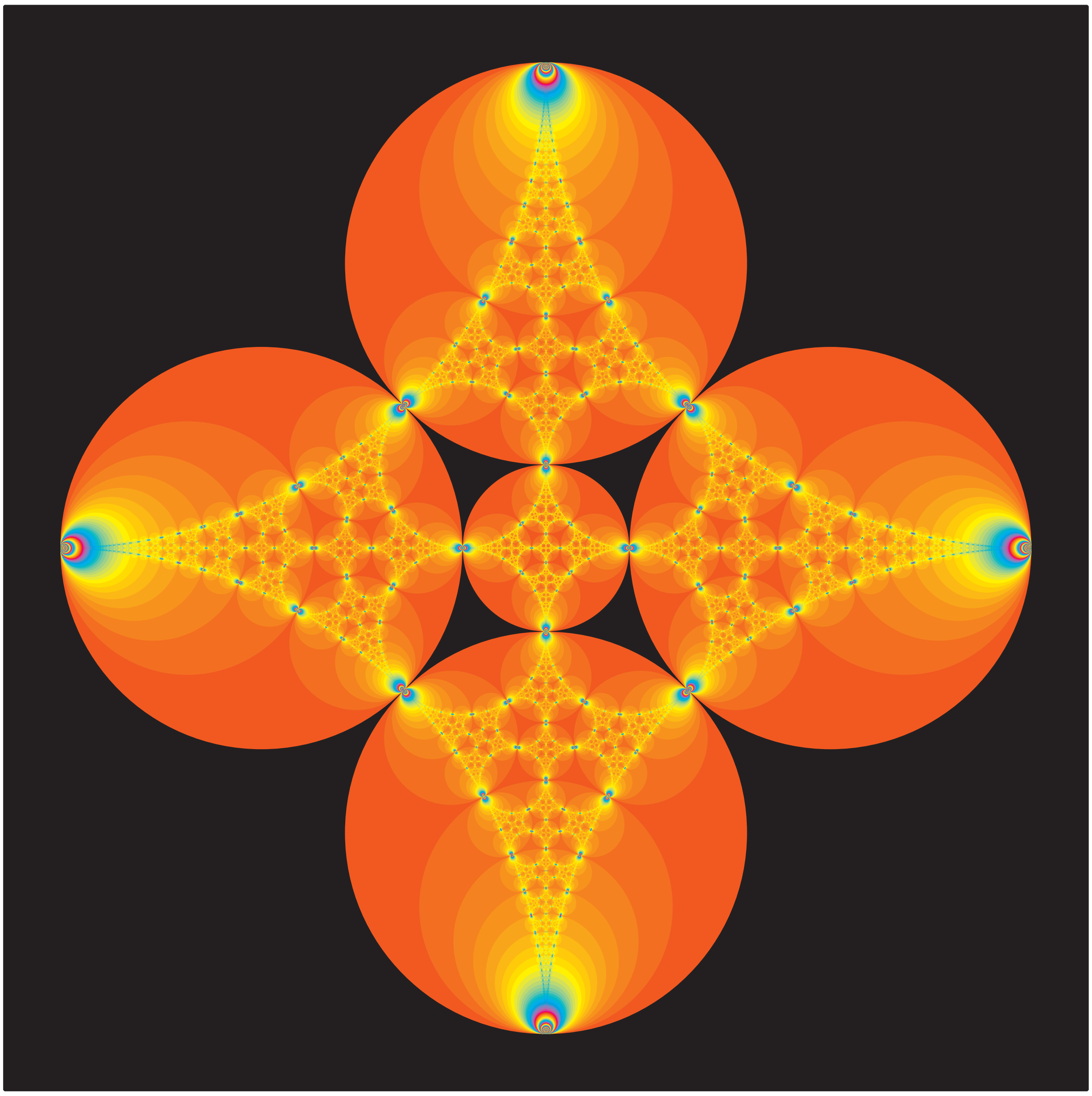
Klein's tilings were now seen to have intimate connections with modern ideas about self-similar scaling behaviour, ideas which had their origin in statistical mechanics, phase transitions and the study of turbulence. There, the self-similarity involved random perturbations, but in Klein's work, one finds self-similarity obeying precise and simple laws.

Strangely, this *exact* self-similarity evokes another link, this time with the ancient metaphor of Indra's net which pervades the *Avatamsaka* or *Hua-yen Sutra*, called in English the *Flower Garland Scripture*, one of the most rich and elaborate texts of East Asian Buddhism. We are indirectly indebted to Michael Berry for making this connection: it was in one of his papers about chaos that we first found the reference from the Sutra to Indra's pearls. Just as in our frontispiece, the pearls in the net reflect each other, the reflections themselves containing not merely the other pearls but also the reflections of the other pearls. In fact the entire universe is to be found not only in each pearl, but also in each reflection in each pearl, and so *ad infinitum*.

As we investigated further, we found that Klein's entire mathematical set up of the same structures being repeated infinitely within each other at ever diminishing scales finds a remarkable parallel in the philosophy and imagery of the Sutra. As F. Cook says in his book *Hua-yen: The Jewel Net of Indra*:

> The Hua-yen school has been fond of this mirage, mentioned many times in its literature, because it symbolises a cosmos in which there is an infinitely repeated interrelationship among all the members of the cosmos. This relationship is said to be one of simultaneous *mutual identity* and *mutual intercausality*.

The Glowing Limit. This illustration follows the mantra of Indra's Pearls *ad infinitum* (at least in so far as a computer will allow). The glowing yellow lacework manifests entirely of its own accord out of our initial arrangement of just five touching red circles.



In the words of Sir Charles Eliot:

> In the same way each object in the world is not merely itself but involves every other object and in fact *is* everything else.

Making a statement equally faithful to both mathematics and religion, we can say that each part of our pictures contains within itself the essence of the whole.

Perhaps we have been carried away with this analogy in our picture *The Glowing Limit* in which the colours have been chosen so that the cluster points of the minutest tiles light up with a mysterious glow. Making manifest of the philosophy of the Sutra, zooming in to any depth (as you will be able to do given your own system to make the programs), you will see the same lace-like structure repeating at finer and finer levels, worlds within worlds within worlds. The glowing pattern is a 'fractal', called the 'limit set' of one of Klein's symmetrical iterative procedures. How to understand and draw such limit sets is what this book is all about.

Like many mathematicians, we have frequently felt frustration at the difficulty of conveying the excitement, challenge and creativity involved in what we do. We hope that this book may in some small way help to redress this balance. On whatever level you choose to read it, be it leafing through for the pictures, reading it casually, playing with the algebra, or erecting a computer laboratory of your own, we shall have succeeded if we have conveyed something of the beauty and fascination of exploring this God-given yet man-made universe which is mathematics.