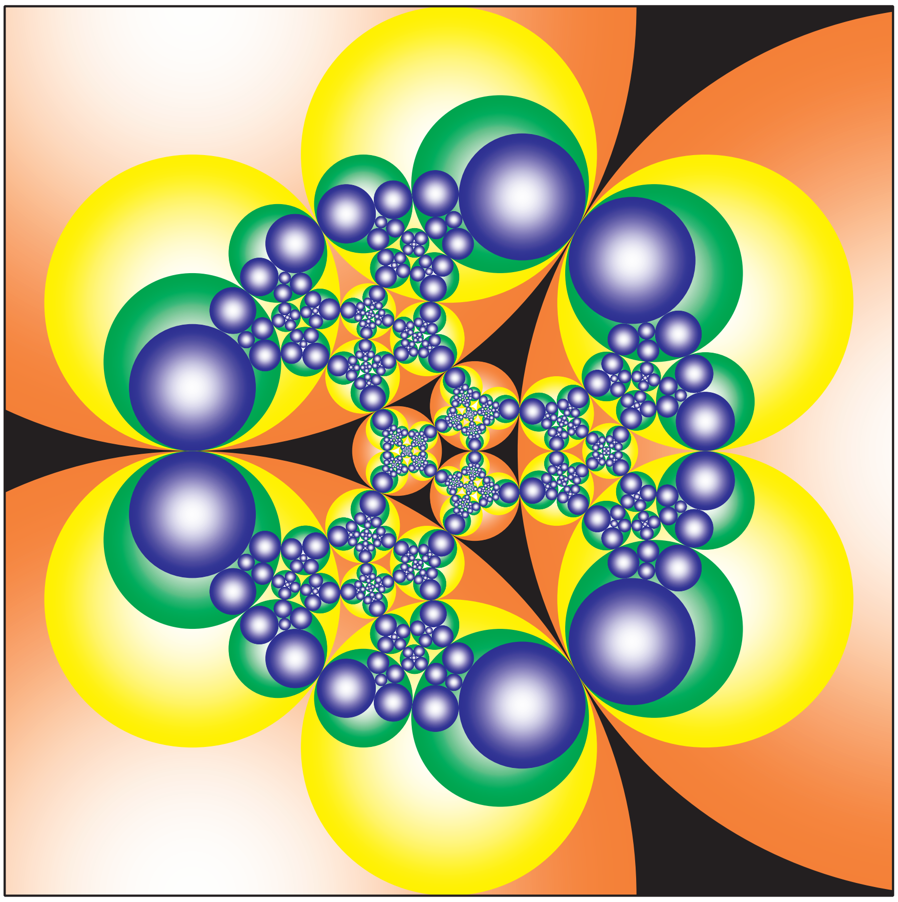
INDRA'S PEARLS

THE VISION OF FELIX KLEIN



The ancient Buddhist dream of Indra's Net

In the heaven of the great god Indra is said to be a vast and shimmering net, finer than a spider's web, stretching to the outermost reaches of space. Strung at each intersection of its diaphanous threads is a reflecting pearl. Since the net is infinite in extent, the pearls are infinite in number. In the glistening surface of each pearl are reflected all the other pearls, even those in the furthest corners of the heavens. In each reflection, again are reflected all the infinitely many other pearls, so that by this process, reflections of reflections continue without end.

Cover picture: A mathematically generated picture foretold in the Buddhist myth of Indra's net? We sometimes call these *Klein Bubbles*. The smallest ones are *sehr klein*.

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The Vision of Felix Klein

David Mumford, Caroline Series and David Wright

With cartoons by Larry Gonick

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PREFACE

What kind of a book is this?

This is a book about serious mathematics, but one which we hope will be enjoyed by as wide an audience as possible. It is the story of our computer aided explorations of a family of unusually symmetrical shapes, which arise when two spiral motions of a very special kind are allowed to interact. These shapes display intricate 'fractal' complexity on every scale from very large to very small. Their visualisation forms part of a century-old dream conceived by the great German geometer Felix Klein. Sometimes the interaction of the two spiral motions is quite regular and harmonious, sometimes it is total disorder and sometimes – and this is the most intriguing case – it has layer upon layer of structure teetering on the very brink of chaos.

As we progressed in our explorations, the pictures that our computer programs produced were so striking that we wanted to tell our tale in a manner which could be appreciated beyond the narrow confines of a small circle of specialists. You can get a foretaste of their variety by taking a look at the Road Map on the final page. Mathematicians often use the word 'beautiful' in talking about their proofs and ideas, but in this case our judgment has been confirmed by a number of unbiassed and definitely non-mathematical people. The visual beauty of the pictures is a veneer which covers a core of important and elegant mathematical ideas; it has been our aspiration to convey some of this inner aesthetics as well. There is no religion in our book but we were amazed at how our mathematical constructions echoed the ancient Buddhist metaphor of Indra's net, spontaneously creating reflections within reflections, worlds without end.

Most mathematics is accessible, as it were, only by crawling through a long tunnel in which you laboriously build up your vocabulary and skills as you abstract your understanding of the world. The mathematics behind our pictures, though, turned out not to need too much in the way of preliminaries. So long as you can handle high school algebra with confidence, we hope everything we say is understandable. Indeed given time and patience, you should be able to make programs to create new pictures for yourself. And if not, then browsing through the figures alone should give a sense of our journey. Our dream is that this book will reveal to our readers that mathematics is not alien and remote but just a very human exploration of the patterns of the world, one which thrives on play and surprise and beauty.

And how did we come to write it?

David M.'s story. This book has been over twenty years in the writing. The project began when Benoit Mandelbrot visited Harvard in 1979/80, in the midst of his explorations of complex iteration – the 'fractals' known as Julia sets – and the now famous 'Mandelbrot Set'. He had also looked at some nineteenth century figures produced by infinite repetitions of simple reflections in circles, a prototypical example of which had fascinated Felix Klein. David W. and I pooled our expertise and began to develop these ideas further in the Kleinian context. The computer rapidly began producing pictures like the ones you will find throughout the book.

What to do with the pictures? Two thoughts surfaced: the first was that they were unpublishable in the standard way. There were no theorems, only very suggestive pictures. They furnished convincing evidence for many conjectures and lures to further exploration, but theorems were the coin of the realm and the conventions of that day dictated that journals only publish theorems.[1] The second thought was equally daunting: here was a piece of real mathematics that we could explain to our non-mathematical friends. This dangerous temptation prevailed, but it turned out to be *much, much* more difficult than we imagined.

[1] Since then, the pioneering team of Klaus and Alice Peters have started the journal *Experimental Mathematics*.

We persevered off and on for a decade. One thing held us back: whenever we got together, it was so much more fun to produce more figures than to write what Dave W. named in his computer `TheBook`. I have fond memories of traipsing through sub-zero degree gales to the bunker-like supercomputer in Minneapolis to push our calculations still further. The one loyal believer in the project was our ever-faithful and patient editor, David Tranah. However, things finally took off when Caroline was recruited a bit more than a decade ago. It took a while to learn how to write together, not to mention spanning the gulfs between our three warring operating systems. But our publisher, our families and our friends told us in the end that enough was enough.

You know that 'word problem' you hated the most in elementary school? The one about ditch diggers. Ben digs a ditch in 4 hours, Ned in 5 and Ted in 6. How long do they take to dig it together? The textbook will tell you 1 hour, 37 minutes and 17 seconds. Baloney! We have uncovered incontrovertable evidence that the right answer is (4+5+6)=15 hours. This is a deep principle involving not merely mathematics but sociology, psychology, and economics. We have a remarkable proof of this but even Cambridge University Press's generous margin allowance is too small to contain it.

David W.'s story. This is a book of a thousand beginnings and for a long time apparently no end. For me, though, the first beginning was in 1979 when my friend and fellow grad student at Harvard Mike Stillman told me about a problem that his teacher David Mumford had described to him: Take two very simple transformations of the plane and apply all possible combinations of these transformations to a point in the plane. What does the resulting collection of points look like?

Of course, the thing was not just to think about the shapes but to actually draw them with the computer. Mike knew I was interested in discrete groups, and we shared a common interest in programming. Also, thanks to another friend and grad student Max Benson, I was alerted to a very nice C library for drawing on the classic Tektronix 4014 graphics terminal. The only missing ingredient was happily filled by a curious feature of a Harvard education: I had passed my qualifying exams, and then I had nothing else to do except write my doctoral thesis. I have a very distinct memory of feeling like I had a lot of time on my hands. As time has passed, I have been astonished to discover that that was the last time I felt that way.

Anyway, as a complete lark, I tagged along with David M. while he built a laboratory of computer programs to visualize Kleinian groups. It was a mathematical joy-ride. As it so happened, in the summer of 1980, there was a great opportunity to share the results of these computer explorations with the world at the historic Bowdoin College conference in which Thurston presented his revolutionary results in three-dimensional topology and hyperbolic geometry. We arranged for a Tektronix terminal to be set up in Maine, and together with an acoustically coupled modem at the blazing speed of 300 bits per second displayed several limit sets. The reaction to the limit curves wiggling their way across the screen was very positive, and several mathematicians there also undertook the construction of various computer programs to study different aspects of Kleinian groups.

That left us with the task of writing an explanation of our algorithms and computations. However, at that point it was certainly past time for me to complete my thesis. Around 1981, I had the very good fortune of chatting with a new grad student at Harvard by the name of Curt McMullen who had intimate knowledge of the computer systems at the Thomas J. Watson Research Center of IBM, thanks to summer positions there. After roping Curt in, and at the invitation and encouragement of Benoit Mandelbrot, Curt and David M. made a set of extremely high quality and beautiful black-and-white graphics of limit sets. I would like to express my gratitude for Curt's efforts of that time and his friendship over the years; he has had a deep influence on my own efforts on the project.

Unfortunately, as we moved on to new and separate institutions, with varying computing facilities, it was difficult to maintain the programs and energy to pursue this project. I would like to acknowledge the encouragement I received from many people including my friend Bill Goldman while we were at M.I.T., Peter Tatian and James Russell, who worked with me while they were undergraduates at M.I.T., Al Marden and the staff of the Geometry Center, Charles Matthews, who worked with me at Oklahoma State, and many other mathematicians in the Kleinian groups community. I would also like to thank Jim Cogdell and the Southwestern Bell Foundation for some financial support in the final stages. The serious and final beginning of this book took place when Caroline agreed to contribute her own substantial research work in this area and her expository gifts, and also step into the middle between the first and third authors to at least moderate their tendency to keep programming during our sporadic meetings to find the next cool picture. At last, we actually wrote some text.

We have witnessed a revolution in computing and graphics during the years of this project, and it has been difficult to keep pace. I would also like to thank the community of programmers around the world for creating such wonderful free software such as TEX, Gnu Emacs, X Windows and Linux, without which it would have been impossible to bring this project to its current end.

During the years of this project, the most momentous endings and beginnings of my life have happened, including the loss of my mother Elizabeth, my father William, and my grandmother and family's matriarch Elizabeth, as well as the birth of my daughters Julie and Alexandra. I offer my part in these pictures and text in the hope of new beginnings for those who share our enjoyment of the human mind's beautiful capacity to puzzle through things. Programming these ideas is both vexing and immensely fun. Every little twiddle brings something fascinating to think about. But for now I'll end.

Caroline's story. I first saw some of David M. and David W.'s pictures in the mid-80s, purloined by my colleague David Epstein on one of his