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The modular group

1 The symplectic group

The symplectic group over $\mathbb{R}$ is a subgroup of the general linear group, defined by certain algebraic equations, and appears from there as an algebraic group.

Definition 1

Let n be a positive integer. The symplectic group of degree n over $\mathbb{R}$ is the subgroup

$$Sp(n, \mathbb{R}) = \{m \in GL(2n, \mathbb{R}) \mid j[m] = j\}, \qquad j = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

Here j is decomposed into $n \times n$ blocks, consisting of the identity matrix $\mathbf{1}$ and the zero matrix 0, respectively. The brackets [ ] always mean the transformation $a[b] := {}^t bab$. The set $Sp(n, \mathbb{R})$ is closed with respect to the group operations in $GL(2n, \mathbb{R})$, since clearly $\mathbf{1} \in Sp(n, \mathbb{R})$, $j[m] = j$ is equivalent to $j = j[m^{-1}]$ by transforming with m^{-1}, and finally $j[m_\nu] = j$, $\nu = 1, 2$, imply

$$j[m_1 m_2] = j[m_1][m_2] = j[m_2] = j.$$

Decomposing m into $n \times n$ blocks,

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the condition $j[m] = j$ is seen to be equivalent to

$${}^t ac,\ {}^t bd \quad \text{symmetric}, \qquad {}^t ad - {}^t cb = \mathbf{1}$$

or, evaluating $j[{}^t m] = j$, to

$$a\,{}^t b,\ c\,{}^t d \quad \text{symmetric}, \qquad a\,{}^t d - b\,{}^t c = \mathbf{1}.$$

Both conditions separately characterize the elements m of $Sp(n, \mathbb{R})$ completely. As a consequence we obtain the following formula for the inverse of a symplectic matrix:

$$m^{-1} = \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}.$$

For $n = 1$, symplecticity just means $\det m = 1$. In general, $\det m = \pm 1$ can be derived from the definitions for arbitrary $m \in Sp(n, \mathbb{R})$. In fact $\det m = 1$ is true, but the converse obviously does not hold for $n > 1$.

It is well known how $Sp(1, \mathbb{R})$ acts on the upper half-plane as a group of biholomorphic mappings. To generalize this subject to arbitrary n, Siegel's half-space of degree n will be introduced.

Definition 2

Let n be a positive integer. Siegel's half-space of degree n consists of all n-rowed complex symmetric matrices z, the imaginary part of which is positive definite,

$$H_n = \{z = x + iy \,|\, {}^t z = z, y > 0\}.$$

Considering the independent entries z_{kl} $(k \leq l)$ of z as coordinates, H_n becomes an open subset of $\mathbb{C}^{n(n+1)/2}$. Since for positive matrices y_1, y_2 and real λ,

$$\lambda y_1 + (1 - \lambda) y_2 > 0 \qquad (0 \leq \lambda \leq 1),$$

H_n turns out to be a convex domain in $\mathbb{C}^{n(n+1)/2}$. As a convex domain, H_n in particular is simply connected. Now we have to use the concept of a holomorphic function in several complex variables for the first time. Following K. Weierstraß we may introduce holomorphic functions as complex-valued functions which can be represented locally by power series expansions. Holomorphic mappings between two domains embedded in complex number spaces are described by holomorphic coordinate-functions. Then we may state

Proposition 1

$Sp(n, \mathbb{R})$ acts on H_n as a group of biholomorphic automorphisms by

$$Sp(n, \mathbb{R}) \times H_n \to H_n, \qquad (m, z) \mapsto m\langle z \rangle := (az + b)(cz + d)^{-1}.$$

Proof

First we show that the matrix $cz + d$ is non-singular. Let $z \in H_n$, $m \in Sp(n, \mathbb{R})$ and put

$$p = az + b, \qquad q = cz + d.$$

Using the symplecticity of m we obtain

$$\begin{aligned} {}^t p \bar{q} - {}^t q \bar{p} &= (z\,{}^t a + {}^t b)(c\bar{z} + d) - (z\,{}^t c + {}^t d)(a\bar{z} + b) \\ &= z - \bar{z} = 2iy. \end{aligned} \tag{1}$$

Now let ξ be any n-rowed complex column such that $q\xi = 0$. Then ${}^t\xi y \bar{\xi} = 0$ by the equation above, hence $\xi = 0$ since y is positive. The system of linear equations $q\xi = 0$ has only the trivial solution; therefore q is non-singular and $m\langle z\rangle$ is well defined. Next, the symmetry of $z^* = m\langle z\rangle$ is equivalent to

$$ {}^t pq = ({}^t z\, {}^t a + {}^t b)(cz + d) = ({}^t z\, {}^t c + {}^t d)(az + b) = {}^t qp, $$

which follows again from the symplecticity of m and the symmetry of z. Now we infer from (1) for the imaginary part of z^*

$$ y^* = \frac{1}{2\mathrm{i}}(z^* - \bar{z}^*) = \frac{1}{2\mathrm{i}}({}^t p\bar{q} - {}^t q\bar{p})\{\bar{q}^{-1}\} = y\{\bar{q}^{-1}\} > 0. $$

Here and from now on we use the brackets $\{\ \}$ to denote the transformation as a Hermitian form, i.e. $a\{b\} := {}^t\bar{b}ab$. So $z^* \in H_n$ and the map in question is of the indicated kind. It is an action of $Sp(n, \mathbb{R})$ on H_n, since one verifies immediately that

$$ (m_1 m_2)\langle z\rangle = m_1\langle m_2\langle z\rangle\rangle, \qquad e\langle z\rangle = z $$

for $m_1, m_2 \in Sp(n, \mathbb{R})$, $z \in H_n$ and the unit element e of $Sp(n, \mathbb{R})$. As with every action of a group on a set, the corresponding mappings

$$ H_n \to H_n, \qquad z \mapsto m\langle z\rangle \tag{2} $$

for fixed $m \in Sp(n, \mathbb{R})$ are bijective. These 'symplectic maps' are holomorphic, since they are rational, and biholomorphic, as the inverse map is performed with m^{-1}. Of course, we could as well have used the fact that every holomorphic bijective map of a domain onto itself is biholomorphic.

If we assign to each $m \in Sp(n, \mathbb{R})$ the automorphism (2) of H_n, we obtain a group homomorphism

$$ Sp(n, \mathbb{R}) \to \mathrm{Bihol}(H_n) $$

of the symplectic group into the group of biholomorphic automorphisms of H_n. The kernel of this homomorphism consists of $\pm\mathbf{1}$, since the identity

$$ m\langle z\rangle = z $$

in H_n implies

$$ az + b = z(cz + d) $$

for arbitrary symmetric z, hence $a = d = \pm\mathbf{1}$, $b = c = 0$. Even the surjectivity of the homomorphism can be shown [64] using a generalization of Schwarz's lemma. So the analogy with the one-variable case is remarkable.

Nevertheless, for certain purposes it is useful to have available another realization of H_n, namely as a bounded symmetric domain in the sense of E. Cartan. This concept was created in order to guarantee the existence of enough non-trivial biholomorphic automorphisms of a domain. Note that in several complex variables there exist domains for which the identity is the only biholomorphic automorphism. Such a domain would be of no use in the theory of automorphic functions. A domain is called homogeneous if the group of biholomorphic automorphisms acts transitively; it is called symmetric if to each point there exists an involution in the group of biholomorphic automorphisms with the given point as a single fixed point. E. Cartan [13] proved that each bounded symmetric domain is homogeneous. I.I. Pjateckij-Šapiro [55] discovered the first example of a non-symmetric but homogeneous bounded domain in 1959. Bounded symmetric domains were classified by E. Cartan [13]; the larger class of bounded homogeneous domains was investigated by I.M. Gelfand, S.G. Gindekin, I.I. Pjateckij-Šapiro and E.B. Vinberg. E. Cartan obtained four main types of irreducible bounded symmetric domains and two exceptional ones that appear only for dimensions 16 and 27, respectively. Here irreducible means that the domain cannot be decomposed into the product of two domains of the same kind. One of Cartan's main types is relevant for our considerations, it is a generalization of the unit-circle to several complex variables.

Definition 3

Let n be a positive integer. The unit-circle of degree n consists of all n-rowed complex symmetric matrices w, for which the Hermitian matrix $\mathbf{1}$-$w\bar{w}$ is positive definite,

$$D_n = \{w \mid {}^t w = w, \mathbf{1} - w\bar{w} > 0\}.$$

Clearly D_n is a bounded domain in $\mathbb{C}^{n(n+1)/2}$. It is related to H_n by a generalized Cayley transformation.

Proposition 2

The Cayley transformation

$$l\colon H_n \to D_n, \qquad z \mapsto w := l\langle z\rangle = (z - i\mathbf{1})(z + i\mathbf{1})^{-1}$$

maps H_n biholomorphically onto D_n.

Proof

For $z \in H_n$ we know that $\det z \neq 0$ (a special case of $\det(cz + d) \neq 0$ in Proposition 1). Since $z + i\mathbf{1} \in H_n$, the Cayley transformation is well defined. Now the symmetry of z implies the symmetry of w and

$$\begin{aligned}\mathbf{1} - w\bar{w} &= \mathbf{1} - (z + \mathrm{i}\mathbf{1})^{-1}(z - \mathrm{i}\mathbf{1})(\bar{z} + \mathrm{i}\mathbf{1})(\bar{z} - \mathrm{i}\mathbf{1})^{-1} \\ &= ((z + \mathrm{i}\mathbf{1})(\bar{z} - \mathrm{i}\mathbf{1}) - (z - \mathrm{i}\mathbf{1})(\bar{z} + \mathrm{i}\mathbf{1}))\{(\bar{z} - \mathrm{i}\mathbf{1})^{-1}\} \\ &= 4y\{(\bar{z} - \mathrm{i}\mathbf{1})^{-1}\} > 0.\end{aligned}$$

Thus the Cayley transformation maps H_n into D_n. On the other hand, $\mathbf{1} - w$ is non-singular for arbitrary $w \in D_n$, for $(\mathbf{1} - w)\xi = 0$ implies $(\mathbf{1} - w\bar{w})\{\bar{\xi}\} = 0$, hence $\xi = 0$. We may therefore form

$$z = \mathrm{i}(\mathbf{1} + w)(\mathbf{1} - w)^{-1} \tag{3}$$

for $w \in D_n$. The symmetry of w implies the symmetry of z; furthermore

$$\begin{aligned}y &= \tfrac{1}{2}((\mathbf{1} - w)^{-1}(\mathbf{1} + w) + (\mathbf{1} + \bar{w})(\mathbf{1} - \bar{w})^{-1}) \\ &= \tfrac{1}{2}((\mathbf{1} + w)(\mathbf{1} - \bar{w}) + (\mathbf{1} - w)(\mathbf{1} + \bar{w}))\{(\mathbf{1} - \bar{w})^{-1}\} \\ &= (\mathbf{1} - w\bar{w})\{(\mathbf{1} - \bar{w})^{-1}\} > 0.\end{aligned}$$

Therefore D_n is mapped into H_n by (3). Obviously the Cayley transformation and (3) are inverse to each other, hence the proposition is proved.

If we introduce the $2n \times 2n$ matrix

$$l = \begin{pmatrix} \mathbf{1} & -\mathrm{i}\mathbf{1} \\ \mathbf{1} & \mathrm{i}\mathbf{1} \end{pmatrix}$$

associated with the Cayley transformation, then the subgroup

$$\Phi_n = l\, Sp(n, \mathbb{R}) l^{-1}$$

of $GL(2n, \mathbb{C})$ acts in the same manner on the unit-circle D_n as $Sp(n, \mathbb{R})$ did on the half-space H_n. Let us determine explicitly the conditions that characterize Φ_n as a subgroup of $GL(2n, \mathbb{C})$. For $m \in GL(2n, \mathbb{C})$ we may state the equivalence

$$m \in Sp(n, \mathbb{R}) \quad \Leftrightarrow \quad j\{m\} = j, \qquad m = \bar{m}.$$

Now we put $m^* = lml^{-1}$ and consider each condition on the right separately:

$$j\{m\} = j \quad \Leftrightarrow \quad j\{l^{-1}\}\{m^*\} = j\{l^{-1}\} \quad \Leftrightarrow \quad k\{m^*\} = k$$

with

$$k = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

and

$$m = \bar{m} \quad \Leftrightarrow \quad m^* l\bar{l}^{-1} = l\bar{l}^{-1}\bar{m}^* \quad \Leftrightarrow \quad c^* = \bar{b}^*, d^* = \bar{a}^*$$

with

$$m^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}.$$

Therefore we obtain

$$\Phi_n = \left\{ m \in GL(2n, \mathbb{C}) \mid m = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, k\{m\} = k \right\}. \tag{4}$$

As an illustration we verify directly

Proposition 3

$Sp(n, \mathbb{R})$, respectively Φ_n, act transitively on H_n, respectively D_n. Both domains are symmetric in the sense of E. Cartan.

Proof

Because of Proposition 2 it is sufficient to consider the action of Φ_n on D_n. First we show its transitivity. Let w be an arbitrary point of D_n; we have to look for an element $m \in \Phi_n$ which transforms w into any distinguished point, for instance the origin. Since $\mathbf{1} - w\bar{w}$ is positive, there exists an $n \times n$ matrix a with complex entries such that

$$(\mathbf{1} - w\bar{w})\{a\} = \mathbf{1}.$$

Put

$$m = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad b = w\bar{a}.$$

Then we obtain

$${}^t\bar{a}a - {}^t b\bar{b} = \mathbf{1}, \qquad {}^t\bar{a}b \text{ symmetric},$$

which is equivalent to $k\{m\} = k$. Therefore $m \in \Phi_n$ by (4) and obviously $w = m\langle 0 \rangle$. Concerning the symmetry, note that $w \mapsto -w$ is an involution of D_n which corresponds to an element of Φ_n and has the origin as a single fixed point. Since we already know that Φ_n is transitive we obtain the same property for any other point of D_n.

On this occasion we prove a useful lemma, which allows us to strengthen Proposition 3 as a corollary.

Lemma

Let w be any n-rowed complex symmetric matrix and

$$d = \begin{pmatrix} \lambda_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{1/2} \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hermitian matrix $w\bar{w}$ in any prescribed order. Then there exists a unitary matrix u such that $w = d[u]$.

Proof

The argument becomes a little involved since transformations as quadratic forms (denoted by [ ]) and as Hermitian forms (denoted by { }) appear at the same time. If w is a diagonal matrix with elements $w_1, \ldots, w_n$ in the main diagonal, we may assume

$$\lambda_\nu = w_\nu \bar{w}_\nu \qquad (\nu = 1, \ldots, n)$$

after reordering, which is an orthogonal transformation. Then we take for u the diagonal matrix formed with the elements

$$\left(\frac{w_\nu}{\bar{w}_\nu}\right)^{1/4} \qquad (\nu = 1, \ldots, n)$$

or 1, if w_ν vanishes. For arbitrary w first determine a unitary matrix u_1 such that $w\bar{w}\{u_1\} = d^2$. Then $q = w[\bar{u}_1]$ is symmetric and $q\bar{q} = d^2$ real. Therefore the real and the imaginary part of q commute and can be transformed into a diagonal form simultaneously by an orthogonal matrix u_2. Then $q[u_2] = w[\bar{u}_1 u_2]$ becomes diagonal too and $\bar{u}_1 u_2$ is unitary. So we have transformed w as a quadratic form into a diagonal matrix by the unitary matrix $u = \bar{u}_1 u_2$. Since such a transformation does not affect the eigenvalues of $w\bar{w}$, we are back to the first case.

Corollary

Let w_1, w_2 be two arbitrary points in D_n. Then there exists an $m \in \Phi_n$ which simultaneously transforms w_1 into 0 and w_2 into a diagonal matrix t, the diagonal elements of which satisfy $0 \leq t_1 \leq \cdots \leq t_n < 1$.

By Proposition 3 we may assume $w_1 = 0$. Take for $t_1, \ldots, t_n$ the square roots of the eigenvalues of $w_2\bar{w}_2$. Then the lemma yields a unitary matrix u such that $w_2 = t[u]$; the mapping $w \mapsto w[u^{-1}]$ is induced by the action of Φ_n and has the desired properties.

The idea of considering the generalized unit-circle instead of Siegel's half-space has two advantages. First, the boundedness is often good for topological considerations, and second the action of Φ_n may be extended holomorphically onto the closure $\bar{D}_n$ of the unit-circle.

Proposition 4

The mappings

$$w \mapsto m\langle w\rangle = (aw + b)(\bar{b}w + \bar{a})^{-1} \qquad (m \in \Phi_n)$$

are topological automorphisms of $\bar{D}_n$. They are holomorphic on a neighborhood of $\bar{D}_n$ (depending on m).

Proof

It is sufficient to verify

$$\det(\bar{b}w + \bar{a}) \neq 0$$

for all $m \in \Phi_n$, $w \in \bar{D}_n$. Now let ξ be a complex column satisfying

$$^t\xi(\bar{b}w + \bar{a}) = 0.$$

Then

$$a\,^t\bar{a}\{\xi\} = \bar{w}w\{^t\bar{b}\xi\}.$$

Note that the condition $k\{m\} = k$ in (4) is equivalent to $k\{^t\bar{m}\} = k$, i.e.

$$a\,^t\bar{a} - b\,^t\bar{b} = \mathbf{1}, \qquad a\,^tb \text{ symmetric.}$$

Thus we obtain

$$(\mathbf{1} + b\,^t\bar{b})\{\xi\} = \bar{w}w\{^t\bar{b}\xi\}$$

or

$$\mathbf{1}\{\xi\} + (\mathbf{1} - \bar{w}w)\{^t\bar{b}\xi\} = 0.$$

Since $\mathbf{1} - \bar{w}w \geq 0$, we infer $\xi = 0$.

Remark

For $Sp(n, \mathbb{R})$ the corresponding proposition is wrong as can be seen, for instance, from the map $z \mapsto -z^{-1}$.

In the remaining part of this section we will mention some basic facts of symplectic geometry which are used later. The well-known model of hyperbolic geometry in the upper half-plane can be generalized to Siegel's half-space H_n by introducing a certain invariant Riemannian metric. The geometrical properties were investigated originally by L.K. Hua, C.L. Siegel and M. Sugawara, and more recently by S. Helgason, A. Korányi, O. Loos, S. Murakami and J. Wolf, amongst others. Differentiating the symplectic mapping

$$z^* = (az + b)(cz + d)^{-1},$$

we obtain

$$dz^*(cz + d) + z^*c\,dz = a\,dz$$

$$dz^*[cz + d] + (z\,^ta + {}^tb)c\,dz = (z\,^tc + {}^td)a\,dz$$

$$dz^*[cz + d] = dz, \tag{5}$$

where $dz = (dz_{kl})$ denotes the matrix of the differentials dz_{kl}. We easily check the transformation law for the imaginary part to be

$$y^* = y\{(cz + d)^{-1}\}. \tag{6}$$

From these two formulas we deduce immediately

$$dz^*y^{*-1}\,d\bar{z}^*y^{*-1} = {}^t(cz + d)^{-1}\,dzy^{-1}\,d\bar{z}y^{-1}\,{}^t(cz + d).$$

Hence the trace $\sigma(\mathrm{d}zy^{-1}\,\mathrm{d}\bar{z}y^{-1})$ defines a quadratic differential form in $\mathrm{d}x_{kl}$, $\mathrm{d}y_{kl}$ $(k \le l)$ over $\mathbb{R}$, which is invariant with respect to the action of $Sp(n, \mathbb{R})$. Since this action is transitive by Proposition 3, we may check the positivity of the differential form at a single point, for instance $z = \mathrm{i}\mathbf{1}$. But there

$$\sigma(\mathrm{d}z\,\mathrm{d}\bar{z}) = \sum_k (\mathrm{d}x_{kk}^2 + \mathrm{d}y_{kk}^2) + 2\sum_{k<l}(\mathrm{d}x_{kl}^2 + \mathrm{d}y_{kl}^2), \tag{7}$$

the positivity of which is obvious. Now we can introduce an invariant Riemannian metric on H_n by the symplectic line element

$$\mathrm{d}s^2 = \sigma(\mathrm{d}zy^{-1}\,\mathrm{d}\bar{z}y^{-1}).$$

Let us transform this line element to the generalized unit-circle by Cayley's transformation. From

$$w = (z - \mathrm{i}\mathbf{1})(z + \mathrm{i}\mathbf{1})^{-1}$$

we deduce

$$\mathrm{d}z = \frac{1}{2\mathrm{i}}\,\mathrm{d}w[z + \mathrm{i}\mathbf{1}].$$

We have already used in the proof of Proposition 2 that

$$y = \tfrac{1}{4}(\mathbf{1} - w\bar{w})\{\bar{z} - \mathrm{i}\mathbf{1}\}.$$

These two formulas allow the calculation of $\mathrm{d}s^2$ in terms of w. The result is

$$\mathrm{d}s^2 = 4\sigma(\mathrm{d}w(\mathbf{1} - \bar{w}w)^{-1}\,\mathrm{d}\bar{w}(\mathbf{1} - w\bar{w})^{-1}).$$

It is certainly interesting to study the Riemannian space H_n equipped with the symplectic metric $\mathrm{d}s^2$ from a geometrical point of view. An excellent survey of the main results concerning geodesics, curvature etc. can be found in Siegel's paper [64]. Since we do not use these geometrical aspects in the following, we restrict ourselves to the determination of the symplectic volume element, which is important in integration theory.

In Riemannian geometry the invariant volume element is defined as the Euclidean volume element multiplied by the square root of the determinant of the quadratic differential form $\mathrm{d}s^2$. Since $Sp(n, \mathbb{R})$ acts transitively on H_n, this volume element is uniquely determined by its invariance up to a constant factor. Therefore it is not necessary to compute the determinant of $\mathrm{d}s^2$ explicitly if any invariant volume element is available by another argument. Then only the determination of an inessential factor remains open. To simplify the computation of Jacobians, note once and for all that the linear map

$$w \mapsto w[c]$$

from the space of n-rowed symmetric matrices w into itself has determinant $\det c^{n+1}$. From this observation all the Jacobians which appear in this book can be read off immediately. So we obtain from (5)

$$\det\left(\frac{\partial z^*}{\partial z}\right) = \det(cz + d)^{-n-1}$$

for the Jacobian of any symplectic map; or, introducing real coordinates,

$$\det\left(\frac{\partial(x^*, y^*)}{\partial(x, y)}\right) = \det\left(\frac{\partial(z^*, \bar{z}^*)}{\partial(z, \bar{z})}\right) = |\det(cz + d)|^{-2n-2}.$$

By (6) we have

$$\left(\frac{\det y^*}{\det y}\right)^{n+1} = |\det(cz + d)|^{-2n-2}.$$

So, if $\mathrm{d}x\,\mathrm{d}y = \prod_{k \le l} \mathrm{d}x_{kl}\,\mathrm{d}y_{kl}$ denotes the Euclidean volume element, then

$$\mathrm{d}v_n = \frac{\mathrm{d}x\,\mathrm{d}y}{\det y^{n+1}}$$

is invariant with respect to the action of $Sp(n, \mathbb{R})$. The volume element of the Riemannian metric $\mathrm{d}s^2$ differs from $\mathrm{d}v_n$ only by a constant factor, which is $2^{n(n-1)/2}$, as can be seen from (7). We call $\mathrm{d}v_n$ the symplectic volume element. A straightforward computation yields

$$\mathrm{d}v_n = 2^{n(n+1)} \frac{\mathrm{d}u\,\mathrm{d}v}{\det(1 - w\bar{w})^{n+1}} \qquad (w = u + iv)$$

for the volume element, after being carried over to the generalized unit-circle D_n by Cayley's transformation.

2 Minkowski's reduction theory

The imaginary parts of the points $z \in H_n$ form the subspace

$$P_n = \{y \mid y > 0\}$$

of $\mathbb{R}^{n(n+1)/2}$ consisting of all n-rowed positive definite matrices with real entries. It is an open convex subset of $\mathbb{R}^{n(n+1)/2}$. Moreover the ray originating from the point 0 and passing through any point $y \in P_n$ lies completely in P_n. Therefore P_n is a convex cone with vertex at the origin. Consider on the other hand the subgroup

$$\left\{m \in Sp(n, \mathbb{R}) \mid m = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, a \in GL(n, \mathbb{R})\right\}$$

of $Sp(n, \mathbb{R})$, which is canonically isomorphic to $GL(n, \mathbb{R})$. Then the action of $Sp(n, \mathbb{R})$ on H_n in Proposition 1.1 induces the action

$$GL(n, \mathbb{R}) \times P_n \to P_n, \qquad (a, y) \mapsto y[{}^t a]$$