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Equivariant cohomology of G-CW-complexes and the Borel construction

By analogy with the notion of a (co)homology theory on some category of topological spaces one can define the notion of an equivariant or G-(co)homology theory on G-spaces, where G is a compact Lie group. Depending on the given frame and intended purpose one actually might impose different sets of axioms for such a definition (see, e.g., [Bredon, 1967b], [tom Dieck, 1987], [Lee, 1968], [May, 1982], [Seymour, 1982]), but the minimal request would be the G-homotopy invariance of the (co)homology functor and a suitable Mayer-Vietoris long exact sequence. These two requirements suffice to get an elementary comparison theorem for G-(co)homology theories similar to the usual non-equivariant case; i.e. if $\tau : h_G \to k_G$ is a natural transformation between G-(co)homology theories, which is an isomorphism on 'G-points' (i.e. homogeneous spaces G/K, K a closed subgroup of G) then $\tau(X)$ is an isomorphism for all G-spaces X, which can be obtained from (finitely many) 'G-points' by a finite number of the following steps (in any order):

(1) replacing a G-space by a G-homotopy equivalent G-space;
(2) taking finite coproducts (topological sums) of G-spaces;
(3) taking homotopy pushouts (double mapping cylinders of G-maps between G-spaces);

(see [Seymour, 1982] for more sophisticated versions of the comparison theorem). The category obtained this way is just the category of G-spaces, which are G-homotopy equivalent to finite G-CW-complexes (cf. [Puppe, D., 1983] for a discussion of this and related questions in the non-equivariant case, not restricted to finite CW-complexes). To extend the comparison theorem to arbitrary G-CW-complexes one needs further additivity properties of the (co)homology theories involved. It is important to realize that these additional properties are not always fulfilled for those G-theories in which we are interested.

The G-theories we are going to consider are variants of the 'Borel (co)homology' of a G-space X, which is the ordinary (co)homology of the Borel construction $EG \times_G X$ on X, where $EG \times_G X$ denotes the total space of the bundle associated with the universal principal G-bundle $G \to EG \to BG$, i.e. the orbit space of $EG \times X$ with respect to the diagonal action (see [Borel *et al.*, 1960]). For G a finite group we will actually use a more algebraic description of this theory, namely as the (co)homology of the group G with coefficients in the (co)chain complex $C(X)$ over $\mathbb{Z}[G]$, where $\mathbb{Z}[G]$ denotes the group ring of G over $\mathbb{Z}$ (see, e.g., [Brown, K.S., 1982]). This provides (co)chain models for the equivariant (co)homology of X, which are very suitable for certain algebraic constructions and for calculations. If G is a p-torus, i.e. an elementary abelian p-group, $G = (\mathbb{Z}_p)^r$, p prime, one can describe the ordinary cohomology of the fixed point set X^K for a subgroup $K < G$ roughly as the Borel cohomology of X with certain coefficients (which depend on K) at least for $p = 2$, the case p an odd prime being somewhat more complicated. In the case of a torus $G = S^1 \times \ldots \times S^1$ the theory of minimal models (see Chapter 2) leads to a formally very similar situation (cf. Section 3.5). This description can be used effectively in place of the Localization Theorem for the Borel cohomology to relate the ordinary cohomology of the G-space X and of the fixed point sets X^K of subgroups $K < G$, which, in fact, is our main theme. The relation to the Localization Theorem is discussed in Chapter 3 in a more general setting.

Other important G-theories like equivariant K-theory (see, e.g. [Segal, 1968a], [Petrie, Randall, 1984]) or equivariant bordism (see, e.g., [Conner, Floyd, 1964]) and cobordism (see, e.g., [tom Dieck, 1970]) will not be considered in this book.

The reader to whom the brief exposition given in the following section, which assumes some familiarity with the very basic definitions and elementary properties of G-spaces, seems unsatisfactory is advised to consult, e.g., [tom Dieck, 1987], Chapter I, as a general reference for basic notions and results in equivariant topology, in particular for the fundamentals in equivariant homotopy theory.

1.1 G-CW-complexes and a comparison theorem for equivariant cohomology theories

The category of G-spaces which are G-homotopy equivalent to G-CW-complexes forms - very similar to the non-equivariant case - a convenient setting for equivariant topology (see [Illman, 1972a, b, 1975], [Matumoto, 1971]). It is closed under homotopy pushouts (double mapping cylinders) and contains, for example, differentiable G-manifolds, which, in fact, admit an actual G-CW-decomposition (see [Illman, 1978, 1983]).

We prove in this section an elementary comparison theorem for equivariant cohomology theories, which provides a useful tool in studying the relations amongst the ordinary cohomologies of the different fixed point sets X^K, $K < G$, where X is a G-space, which is G-homotopy equivalent to a finite (resp. finite-dimensional) G-CW-complex. We will be mainly considering the case where G is a torus or a p-torus (i.e. an elementary abelian p-group, p a prime), but for this section there is no essential difficulty in treating the more general case where G is any compact Lie group and $K < G$ denotes a closed subgroup.

(1.1.1) Definition

(1) By the n-dimensional G-cell of type G/K we mean the G-space $G/K \times D^n$, where G acts by left translation on G/K and trivially on the n-dimensional ball D^n. We call $G/K \times S^{n-1}$ the G-boundary of the G-cell $G/K \times D^n$.

(2) We say that the G-space X is obtained from the G-space Y by attaching a disjoint union $\coprod_\nu G/K_\nu \times D^n$ of n-dimensional G-cells $G/K_\nu \times D^n$, $K_\nu < G$, along the G-maps ϕ_ν: $G/K_\nu \times S^{n-1} \to Y$ if $X = Y \cup_{\phi_\nu} (\coprod_\nu G/K_\nu \times D^n) := Y \sqcup (\coprod_\nu G/K_\nu \times D^n) / \sim$ where $\phi_\nu([g]_\nu, s) \sim ([g]_\nu, s)$ for $([g]_\nu, s) \in G/K_\nu \times S^{n-1}$. (Observe that G-maps ϕ: $G/K \times S^{n-1} \to Y$ correspond one-to-one to (non-equivariant) maps

$$\tilde{\phi} : S^{n-1} \cong \{1\} \times S^{n-1} \to Y^K,$$

($1 =$ unit element of G).) There is an obvious canonical G-inclusion $Y \subset X$.

(3) A G-CW-complex X is a G-space which is obtained as the colimit of a sequence of G-inclusions $X^0 \subset X^1 \subset \ldots \subset X^{n-1} \subset X^n \subset \ldots$, where X^0 is a disjoint union of homogeneous spaces G/K_ν (0-dimensional G-cells) and X^n is obtained from X^{n-1} by attaching a disjoint union of n-dimensional G-cells.

A finite (resp. finite-dimensional) G-CW-complex is, of course, a G-

CW-complex which is constructed from finitely many G-cells (resp. from G-cells of bounded dimension).

It should be noted that one might have to distinguish between the equivariant and the non-equivariant dimension since the dimsension of G/K might be positive, if G is not a finite group.

The orbit space X/G of a G-CW-complex inherits in the obvious way a CW-structure from the G-CW-structure of X. The (non-equivariant) cells of X/G are just the orbit spaces of the G-cells of X and the attaching maps of X/G are those induced from the equivariant maps of X by dividing out the G-action.

The above definition is modelled after the usual definition of CW-complexes, replacing the 'points' by 'G-points', i.e. homogeneous spaces, and many topological and homological properties of CW-complexes carry over or have their 'natural' counterparts in the equivariant setting (see [Bredon, 1967b], [Illman, 1972a, b, 1975], [Matumoto, 1971], [May, 1982], [Seymour, 1982], [tom Dieck, 1987], Lück [1989]). Here we shall only need a few rather simple facts about G-CW-complexes. The main use we are making of them is to get a simple comparison theorem for G-cohomology theories (see Theorem (1.1.3)).

(1.1.2) Examples

(1) For $G = \mathbb{Z}_2$ the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ together with the G-action given by scalar multiplication,

$$G \times S^{n-1} \to S^{n-1}, (\lambda, x) \mapsto \lambda x, \lambda \in G = \{\pm 1\} \subset \mathbb{R}, x \in S^{n-1} \subset \mathbb{R}^n$$

is a free G-space, denoted by X, which has the following decomposition as G-CW-complex:

$$X^0 := G \cong S^0$$
$$X^m := X^{m-1} \cup_{\phi^m} G \times D^m, \text{ where } \phi^m\colon G \times S^{m-1} \to X^{m-1} = S^{m-1}$$
$$\text{is given by } \phi^m(\lambda, x) = \lambda x \ (1 \le m < n).$$

The orbit space X/G is the real projective space $\mathbb{R}P^{n-1}$, and the G-CW-structure of X induces the standard (non-equivariant) cell decomposition of $\mathbb{R}P^{n-1}$.

(2) For $G = S^1$ the unit sphere $S^{2n-1} = \{z \in \mathbb{C}^n : \|z\| = 1\}$ together with the G-action given by complex scalar multiplication

$$G \times S^{2n-1} \to S^{2n-1}, (\lambda, z) \mapsto \lambda z, \lambda \in G = S^1 \subset \mathbb{C}, z \in S^{2n-1} \subset \mathbb{C}^n$$

is a free G-space X with an analogous decomposition as a G-CW-complex:

$$X^0 := G \cong S^1, X^1 := X^0$$

$X^{2m} = X^{2m-1} \cup_{\phi^{2m}} G \times D^{2m}$, where

$$\phi^{2m} : G \times S^{2m-1} \to X^{2m-1} = X^{2m-2} = S^{2m-1}$$

is given by

$$\phi^{2m}(\lambda, z) = \lambda z \quad (1 \leq m < n).$$

(Note that the equivariant $2m$-skeleton X^{2m} of X is not a $2m$-skeleton of a corresponding non-equivariant cell decomposition of X due to the fact that $G = S^1$ has dimension 1.) The orbit space X/G is the complex projective space $\mathbb{C}P^{n-1}$, and again the given G-CW-structure of X induces the standard non-equivariant CW-decomposition of $\mathbb{C}P^{n-1}$.

(3) Restricting the S^1-action on S^{2n-1} in Example (2) to a subgroup $G = \mathbb{Z}_n < S^1$ clearly gives a free G-action on S^{2n-1}. But the S^1-CW-structure of S^{2n-1} does not give a G-CW-structure of S^{2n-1} for $G = \mathbb{Z}_n$ 'on the nose', since $S^1 \times D^{2m}$ is obviously not a G-cell in the sense of Definition (1.1.1). Of course $S^1 \times D^{2m}$ can be decomposed into G-cells and this leads to the following G-CW-structure of the G-space S^{2n-1}

$X^0 := G$

$X^1 := G \cup_{\phi^1} G \times D^1 \cong S^1$,

$\phi^1 : G \times S^0 \to G$ is the unique equivariant extension of

$$\tilde{\phi} : S^0 \to G,$$

$\tilde{\phi}(1) = e$, $\tilde{\phi}(-1) = g \in G$, where g denotes a fixed generator of G

$X^{2m} := X^{2m-1} \cup_{\phi^{2m}} G \times D^{2m}$,

$\phi^{2m} : G \times S^{2m-1} \to X^{2m-1} \cong S^{2m-1}$ is given by $\phi^{2m}(\lambda, x) = \lambda x (1 \leq m < n)$

$X^{2m+1} := X^{2m} \cup_{\phi^{2m+1}} G \times D^{2m+1}$,

$\phi^{2m+1} : G \times S^{2m} \to X^{2m}$ is the unique equivariant extension of $\tilde{\phi}^{2m+1} : S^{2m} \to X^{2m}$, where $\tilde{\phi}^{2m+1}$ is defined as follows: S^{2m} can be written as the union of the right (D^{2m}_+) and the left (D^{2m}_-) hemisphere glued together along the boundary S^{2m-1}, $S^{2m} = D^{2m}_+ \cup_{S^{2m-1}} D^{2m}_-$; and X^{2m} is the quotient of $G \times D^{2m}$ by an equivalence relation given by the attaching map ϕ^{2m}. With this notation we put $\tilde{\phi}^{2m+1}(x_+) = [e, x]$, $x_+ \in D^{2m}_+$ and x the corresponding element in D^{2m}, $\tilde{\phi}^{2m+1}(x_-) = [g, g^{-1}x]$, $x_- \in D^{2m}_-$, $x \in D^{2m}$, g the chosen generator of G. Note that the equivalence relation on $G \times D^{2m+1}$ given by ϕ^{2m+1} is such that the above definition gives a well-defined map $\tilde{\phi}^{2m+1}$: $S^{2m} \to X^{2m}$ ($[e, x] = [g, g^{-1}x]$ in X^{2m} if $x \in S^{2m-1}$).

(4) The universal free G-space EG (contractible, disregarding the G-action), i.e. the total space of the universal principal G-bundle can be considered an (infinite) G-CW-complex. The orbit space $BG := EG/G$, i.e. the classifying space of the group G, inherits a CW-structure from the G-CW-structure of EG.

The space EG is well defined only up to G-homotopy equivalence. For concise treatment of the relevant homotopy theory see, e.g., [tom Dieck, 1987]. For the groups $G = \mathbb{Z}_2$, $\mathbb{Z}_n$ or $G = S^1$ one can just take the colimit (over the canonical inclusions given by the G-CW-structure) of the free G-space in (1), (3) or (2), respectively, to get a G-CW-structure on $EG = \operatorname{colim} S^n = S^\infty$. (By standard homotopy arguments this colimit is contractible since $S^n \subset S^{n+1}$ is a cofibration and is homotopic to the constant map.) Note that in case $G = \mathbb{Z}_n$ (including $n = 2$), the cellular chain complex $W_*(EG)$ considered as a complex over the group ring $\mathbb{Z}[G]$ together with the natural augmentation $\varepsilon : W_*(EG) \to W_*(pt) = \mathbb{Z}$ is the standard minimal free acyclic resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module, which is used to define the homology (and cohomology) of the group G in a purely algebraic context (see, e.g., [Brown, K.S., 1982]).

To get EG as a G-CW-complex for the groups we mainly consider (i.e. $G = (\mathbb{Z}_p)^r$, p prime, or $G = (S^1)^r = T^r$) one can take the product of the universal free G-spaces of the single factors defining the G-CW-structure (and the 'weak' topology) on the product in a similar way as in the non-equivariant situation.

In general, the following construction gives an explicit G-CW-structure on EG. Note that it is not a generalization of the construction described above, in fact for $G = \mathbb{Z}_p$ the number of cells used in a fixed positive dimension is bigger than in the 'minimal' construction above.

$EG_0 := G$

$EG_n := EG_{n-1} \cup_{\phi_n} G \times (G^n \times \Delta_n)$, where $\Delta_n \cong D^n$ is the standard n-simplex, $\dot{\Delta}_n \cong S^{n-1}$ its boundary and $\phi_n : G \times (G^n \times \dot{\Delta}_n) \to EG_{n-1}$ given by

$$\phi_n(g_0, \ldots, g_n, x_0, \ldots, x_n) := \begin{cases} [g_0, \ldots, g_{n-1}, x_0, \ldots, x_{n-1}] \\ \quad \text{if } x_n = 0 \\ [g_0, \ldots, g_i g_{i+1}, \ldots, g_n, x_0, \\ \quad \ldots, \hat{x}_i, \ldots, x_n] \\ \quad \text{if } x_i = 0 \text{ and } 0 \le i < n. \end{cases}$$

Here $(x_0, \ldots, x_n)$ are the barycentric coordinates of a point $x \in \triangle_n$; $\hat{}$ means omit the corresponding term and $[\ldots]$ denotes the equivalence class in EG_{n-1}. It is straightforward to check that ϕ_n is a well-defined G-map. $EG := \operatorname{colim} EG_n$, where the colimit is taken with respect to the canonical inclusions $EG_{n-1} \subset EG_n$. By construction these inclusions are cofibrations (in fact they are G-cofibrations, i.e. cofibrations in the category of G-spaces). Hence to show that EG is contractible (disregarding the G-action) it suffices to prove that the inclusions $EG_{n-1} \subset EG_n$ are null-homotopic. The map

$$\phi : G \times (G^{n-1} \times \triangle_{n-1}) \times I \to G \times (G^n \times \triangle_n)$$

given by

$$\begin{aligned}\phi(g_0, \ldots, &g_{n-1}, x_0, \ldots, x_{n-1}, t) \\ &= (g_0, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_0^{-1}, (1-t)x_0, \ldots, (1-t)x_{n-1}, t)\end{aligned}$$

induces the desired homotopy between the inclusion $EG_{n-1} \subset EG_n$ and the constant map to $[e] \in EG_0 \subset EG_n$ (here e denotes the unit element in $G = EG_0$) since ϕ is compatible with the necessary identifications to obtain a map from $EG_{n-1} \times I$ to EG_n and

$$\begin{aligned}\phi(g_0, \ldots, g_{n-1}, x_0, \ldots, x_{n-1}, 0) &= [g_0, \ldots, g_{n-1}, x_0, \ldots, x_{n-1}] \\ \phi(g_0, \ldots, g_{n-1}, x_0, \ldots, x_{n-1}, 1) &= [g_0, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_0^{-1}, 0, \ldots 0, 1] \\ &= [e, 1] = [e].\end{aligned}$$

If G is a finite group, then $EG_0 \subset EG_1 \subset \ldots EG_{n-1} \subset \ldots$ together with the attaching maps ϕ_n is a G-CW-structure for EG since $G \times (G^n \times \triangle_n)$ is a disjoint union of n-dimensional G-cells. In case G is not finite (i.e. the dimension of G as a manifold is greater than 0) the attaching of $G \times (G^n \times \triangle_n)$ to EG_{n-1} can be decomposed into successively attaching free G-cells of different dimensions to EG_{n-1} corresponding to a non-equivariant cell decomposition of $G^n \times \triangle_n$.

For a very conceptual construction of EG and its orbit space $BG = EG/G$ from a categorical viewpoint and a discussion of the relation to Milnor's construction of the universal principal G-bundle see [Segal, 1968b].

(5) If X is a (compact) differentiable manifold with a differentiable G-action, then X can be viewed as a (finite) G-CW-complex (see [Illman, 1978, 1983]).

(6) For G a finite group, a (finite) simplicial G-complex in the sense of Bredon [1972] can be viewed - after barycentric subdivision - as a (finite) G-CW-complex.

Many of the basic concepts and results (e.g. Whitehead Theorem, cellular approximation, obstruction theory) carry over to G-CW-complexes (see [May, 1982] for a resumé and the references given there for details). In particular any G-space X can be approximated by a G-CW-complex X_{CW} in the following sense: there exists a G-map

$$\phi : X_{CW} \to X$$

which induces a weak equivalence for the fixed point sets of all closed subgroups $K \subset G$, i.e. $\phi^K\colon X^K_{CW} \to X^K$ induces isomorphisms of the (non-equivariant) homotopy groups. This is quite obvious in the case of a finite group G (using the canonical G-structure on the singular simplicial complex of X and on its realization), but holds, more generally, for any compact Lie group G (see [Waner, 1980], [Seymour, 1983], [Matumoto, 1984]).

In certain situations it is of interest to decide whether a G-space X can be approximated by a finite G-CW-complex in the above sense. This is a very difficult question in general (see, e.g., [Lück, 1989] for a comprehensive treatment of this and related topics). It is shown in [Petrie, Randall, 1986] that a non-singular affine variety, on which a finite group G acts algebraically, has the G-homotopy type of a finite G-CW-complex.

The following result, which is, in fact, a rather straightforward G-analogue of the well-known comparison theorem for (non-equivariant) cohomology theories, will be applied in the next section.

Equivariant cohomology theories are considered on suitable categories of G-spaces or pairs of G-spaces (see, e.g., [Bredon, 1967b], [tom Dieck, 1987], [May, 1982], [Lee, 1968], [Segal, 1968a], [Seymour, 1982]). Although the axioms required for an equivariant cohomology theory depend to some extent on the context, a $\mathbb{Z}$-graded equivariant cohomology theory $h^*_G = \{h^q_G\}_{q\in\mathbb{Z}}$ on the category of (finite) G-CW-complexes taking values in a category of modules over a ring R should always fulfil:

GCT I: h^*_G is G-homotopy invariant, i.e. $h^*_G(f_0) = h^*_G(f_1)$ if $f_0, f_1\colon X \to Y$ are G-homotopic.

GCT II: If X is obtained from X_1 by attaching X_2 along a cellular G-map $\phi\colon X_0 \to X_1$, where X_0 is a subcomplex of X_2, then there is a long exact Mayer-Vietoris sequence

$$\dots h^q_G(X_0) \xrightarrow{\delta} h^{q+1}_G(X) \to h^{q+1}_G(X_1) \times h^{q+1}_G(X_2) \to h^{q+1}_G(X_0) \to \dots$$

The properties GCT I and GCT II suffice to prove the following result:

(1.1.3) Theorem

If $\tau\colon h^*_G \to k^*_G$ is a natural transformation between G-cohomology theories, which fulfil GCT I and GCT II above and if $\tau(G/K)$ is an isomor-

phism for each $K < G$ (i.e. for each 'G-point'), then τ is an isomorphism for each finite G-CW-complex.

Proof.

(a) That $\tau(G/K \times S^{n-1})$ is an isomorphism follows from GCT I and GCT II and the five lemma by decomposing $X = G/K \times S^{n-1} = X_1 \cup_\phi X_2$ with $X_1 = G/K \times D^0$, $X_2 = G/K \times D^{n-1}$ and

$$\phi: \quad X_0 = G/K \times S^{n-2} \to X_1$$

given by collapsing S^{n-2} to a point ($n \geq 1$, $S^{-1} := \emptyset$).

(b) That $\tau(X)$ is an isomorphism for any finite G-CW-complex X now follows by induction on the skeletons of X using again GCT I and GCT II, the five lemma, and part (a). □

(1.1.4) Remarks

(1) If one assumes that the cohomology theories considered in (1.1.3) are strongly additive (i.e. $h^*_G(\coprod_\nu X_\nu) = \prod_\nu h^*_G(X_\nu)$ for any disjoint union $\coprod_\nu X_\nu$, and similarly for k^*_G), then Milnor's $\lim^1$-argument (see [Milnor, 1962]) together with the above gives that $\tau(X)$ is an isomorphism for any G-CW-complex. But although many equivariant cohomology theories are strongly additive, some of those we are going to consider in the next paragraph are not. As we shall point out this reflects the fact that P.A. Smith theory needs some kind of finiteness assumption.

(2) Using the same argument as above one obtains that $\tau(X)$ is an isomorphism for any finite-dimensional G-CW-complex X if $\tau(\coprod_\nu G/K_\nu)$ is an isomorphism for any disjoint union $\coprod_\nu G/K_\nu$ of G-points. If only finitely many orbit types G/K occur in X it suffices that $\tau(\coprod G/K)$ is an isomorphism, where $\coprod G/K$ denotes an arbitrary disjoint union of G-points of the same type G/K (see, e.g., (1.3.5), (1.4.5) and (3.1.6)).

(3) The above results obviously extend to G-spaces that are G-homotopy equivalent to finite, resp. finite-dimensional, G-CW-complexes. This holds, in fact, for almost all the results we are going to present, i.e. the assumption 'X a G-CW-complex' can usually be replaced by 'X G-homotopy equivalent to a G-CW-complex' for obvious reasons.

(1.1.5) Remark

It is left to the interested reader to formulate and prove the analogous results for equivariant homology theories.

For more sophisticated versions of the comparison theorem in the equivariant context see [Seymour, 1982].

1.2 The Borel Construction

A. Borel [Borel *et al.*, 1960] introduced the following method to study the cohomology of G-spaces. It has become a basic tool in the study of transformation groups.

Let X be a G-space and EG the universal free G-space (G a compact Lie group) then $X_G := EG \times_G X$ - the orbit space of the diagonal action on the product $EG \times X$ (where EG and X are assumed to be left G-spaces) - is the total space of the bundle $X \to X_G \to BG$ associated to the universal principle bundle $G \to EG \to BG$ ($BG := EG/G$ the classifying space of G). The cohomology $H^*(X_G)$ of X_G, viewed as a functor in the variable X can be considered as an equivariant cohomology theory in the sense of Section 1.1. For certain groups G, in particular for tori and p-tori, and appropriate coefficients, the $H^*(BG)$-module structure of $H^*(X_G)$, given by the map $H^*(BG) = H^*(*_G) \to H^*(X_G)$ induced by the projection $X_G = EG \times_G X \to EG \times_G * = BG$ ($*$ denotes a one-point space) and by the cup-product, strongly reflects relations between the (ordinary) cohomology of the G-space X and its fixed point set. Most of what we discuss in this book is concerned with exploiting this connection.

In Chapter 1 we restrict ourselves to finite groups if not explicitly noted otherwise; in fact we mainly consider p-tori, and we give a more algebraic description of the above equivariant cohomology which provides certain cochain models that can be used very effectively to study the relation between the cohomology $H^*(X^K)$ of the different fixed-point sets X^K, $K < G$, of the G-space X (including in particular the fixed point set X^G and the whole space $X = X^{\{1\}}$, $\{1\} < G$). For a concise treatment of the fundamental topological properties of the Borel construction see, e.g., [tom Dieck, 1987]. (The reader should realize that the Borel construction is well-defined only up to homotopy, since the universal free G-space is only well-defined up to G-homotopy. But applying the homotopy invariant functor $H^*(-)$ to the Borel construction X_G on the G-space X gives (up to natural equivalence) a well-defined equivariant cohomology theory $H^*_G(X) = H^*(X_G)$.)

Let $\mathcal{E}_*(G) := W_*(EG; k)$ be the cellular chain complex of the universal free G-CW-complex (see Section 1.1) with coefficients in a principal ideal domain k. (We are mainly concerned with $k = \mathbb{Z}$ or k a field and we are going to suppress the coefficients from the notation when there is no danger of confusion. If necessary we will indicate them by writing $\mathcal{E}_*(G; k)$ instead of $\mathcal{E}_*(G)$.) The complex $\mathcal{E}_*(G)$ inherits a G-structure from EG, i.e. can be considered as a complex over the group ring $k[G]$ of