LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

1 Introduction to combinators and λ-calculus, J.R. HINDLEY & J.P. SELDIN
2 Building models by games, WILFRID HODGES
3 Local fields, J.W.S. CASSELS
4 An introduction to twistor theory, S.A. HUGGETT & K.P. TOD
5 Introduction to general relativity, L. HUGHSTON & K.P. TOD
6 Lectures on stochastic analysis: diffusion theory, DANIEL W. STROOCK
7 Dynamical systems and the theory of evolution, J. HOFBAUER & K. SIGMUND
8 Summing and nuclear norms in Banach space theory, G.J.O. JAMESON
9 Automorphisms of surfaces after Nielsen and Thurston, A. CASSON & S. BLEILER

London Mathematical Society Student Texts. 8

Summing and Nuclear Norms in Banach Space Theory

G.J.O. JAMESON
Department of Mathematics, University of Lancaster

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York New Rochelle Melbourne Sydney

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521341349

First published 1987
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-34134-9 hardback
ISBN 978-0-521-34937-6 paperback

To the people of Nicaragua

Cambridge University Press
978-0-521-34937-6 - Summing and Nuclear Norms in Banach Space Theory
G. J. O. Jameson
Frontmatter
More information

CONTENTS

INTRODUCTION

The summing and nuclear norms of linear operators merit recognition as very basic concepts in Banach space theory, even at quite an elementary level. They have powerful applications to a variety of Banach space questions, and they generate a theory that is interesting and elegant in its own right. It is hoped that the pages that follow will go some way towards justifying these assertions. The only prerequisite is a beginner's course on normed linear spaces. As well as the confirmed Banach space specialist, our topic has something to offer to analysts whose main interest is, for example, approximation theory or operator theory.

The origins of the subject can be traced to Khinchin's inequality (published in 1923) and to Orlicz's deduction (1933) that for every unconditionally convergent series Σx_n in L_p (where $1 \leqslant p \leqslant 2$), $\Sigma \|x_n\|^2$ is convergent. In 1947, Macphail showed that in ℓ_1, such a series may have $\Sigma \|x_n\|$ divergent. Dvoretzky and Rogers then proved that the same applies in every infinite-dimensional Banach space. From this, it was a short step to define an "absolutely summing operator" to be one for which $\Sigma \|Tx_n\|$ is convergent for every unconditionally convergent series Σx_n. Further, Macphail's work showed how this property is equivalent to a certain numerical quantity being finite: this is the "1-summing norm" $\pi_1(T)$. The idea generalizes easily to give norms π_p for each finite $p \geqslant 1$. The most interesting, and "natural", cases are $p = 1,2$, and in this book our account will be largely concentrated on these cases. For operators between Hilbert spaces, π_2 coincides with the classical Hilbert-Schmidt norm.

The "nuclear" norms ν_p (for $1 \leqslant p \leqslant \infty$) are dual, in a very natural sense, to the summing norms: $\nu_{p'}$ is dual to π_p, and ν_1 to ordinary operator norm.

It is the norms themselves, rather than the corresponding classes of p-summing and p-nuclear operators, that have proved to be of such value in

Banach space theory: hence our title. In the case of ν_p, it will be enough for our purposes to confine attention to operators of finite rank. There are close connections between these norms and other numerical quantities that are the essential tools of the subject, such as projection constants, Banach-Mazur distances and basis constants. Indeed, for finite-dimensional spaces, the projection constant $\lambda(X)$ is precisely $\nu_\infty(I_X)$. The quantity $\pi_1(I_X)$ can itself be regarded as a constant characteristic of the space. It is so closely related to the projection constant that at times both can be evaluated together. We show how this calculation can be done for the spaces ℓ_1^n and ℓ_2^n.

The central theorem of the subject is Pietsch's theorem on the existence of a dominating functional for p-summing operators. In the case p = 2, this theorem combines with the nice properties of Hilbert spaces to show that (i) every 2-summing operator can be factorized through a Hilbert space, and (ii) 2-summing operators can be extended, with the value of π_2 preserved (in other words, π_2 is the notion that delivers a "Hahn-Banach" theorem for operators).

Pietsch's theorem also provides a beautifully simple proof that the projection constant of an n-dimensional space, and the distance to ℓ_2^n, are not greater than $\sqrt{n}$. This application on its own is perhaps enough to justify the claim that these norms have a rightful place is Banach space theory. Another good application is the Gordon-Lewis proof that the ordinary space of operators on ℓ_2^n has a basis constant that grows with n.

Some of the deepest results in the theory involve the comparison of different summing norms for operators between certain spaces. Theorems of this sort were initiated by Grothendieck, and the most important one is the result known as "Grothendieck's inequality", which is coming to be recognized as one of the really major theorems in Banach space theory. Part of the fascination of this theorem is its abundance of equivalent formulations. It can be (and often is) stated in terms of bilinear forms or tensor products instead of summing norms, and it has important applications in harmonic analysis. This illustrates again how intricately these norms are connected with other topics of established interest. The notions of type 2 and cotype 2 constants are the key to a wider formulation of results of this sort. In particular, the essence of Grothendieck's inequality is generalized by Maurey's theorem stating that all spaces of cotype 2 are "2-dominated".

There is a constant interplay between finite-dimensional and infinite-dimensional spaces. Some of the results are specifically concerned with finite dimensions. Others, including Grothendieck's inequality, apply to infinite-

Cambridge University Press
978-0-521-34937-6 - Summing and Nuclear Norms in Banach Space Theory
G. J. O. Jameson
Frontmatter
More information

dimensional spaces, but with all the real work taking place in a finite-dimensional context. Pietsch's theorem itself does not require any element of finite dimensionality, but (as remarks above show) many of its applications do.

The summing and nuclear norms are examples (arguably the most important ones) of "operator ideal" norms, and thereby provide an introduction to the rapidly growing research area that is becoming known as operator ideals.

Concepts and definitions are introduced gently, with plenty of simple examples (these seem to be almost entirely lacking in the existing literature). Proofs are generally complete, though the details of some of the examples are left to the reader. The author is strongly committed to the principle that proofs should be as simple and direct as possible, and that they should give a "feel" for why a result is true, as well as establishing it formally. A number of the results in this book appear with a proof that is substantially simpler than the original one - though in most cases the author does not claim any of the credit for this. In other instances - such as the derivation of Khinchin's inequality with the best constant - a satisfactorily simple proof is still awaited. There are several instances where two alternative proofs are given, since both contribute something to the understanding. Results and examples are numbered consecutively in each section. Moderately important results are designated "proposition", and the most important ones "theorem". Exercises are scattered through the text, appearing at the point where they are most relevant.

The list of references is intended both to point the way to further reading and to pay some respect to those who have developed the highly satisfying theory presented here. I have endeavoured to give just enough attributions to identify the main landmarks in this development - but this does not amount to an attempt to give a systematic historical survey.

Sections 1 to 11 contain the core material, and need to be read more or less in order. The remaining sections deal with a selection of further topics, and are independent of each other.