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Combinatorial Group Theory: a topological approach

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To the memories of Ralph Fox, from whom I first learned of the connections between group theory and topology, and of Hanna Neumann and Roger Lyndon, whose works extended my interest in group theory.

INTRODUCTION

Combinatorial group theory can be regarded as that branch of group theory which considers groups given by generators and relations. Some of its basic results involve manipulation with words; that is, products of the generators and their inverses. It is this aspect to which the word "combinatorial" refers; it is not connected with that branch of mathematics known as combinatorics.

From its earliest stages this theory has been closely connected with topology. To any topological space there is an associated group, called the *fundamental group* of the space. In trying to investigate properties of certain spaces we are led to problems in combinatorial group theory. Conversely, some problems in combinatorial group theory are best solved by geometric and topological discussions of suitable fundamental groups.

It is this interplay between group theory and topology which is the theme of this book. The texts on combinatorial group theory by Magnus, Karrass, and Solitar (1966) and by Lyndon and Schupp (1977) go much deeper into the group theory, but have little to say about the topology, while the texts by Massey (1967) and Stilwell (1980) concentrate on the topology rather than the group theory.

Chapter 1 contains the main constructions of combinatorial group theory; free groups, presentations, free products, amalgamated free products, and the HNN extension. The various Normal Form Theorems are proved, with several different proofs, and applications of the constructions are made (for instance, to show the existence of a finitely generated infinite simple group).

Chapter 2 has some topological preliminaries; ways of building new spaces from old ones, and a discussion of paths in spaces.

From looking at paths in spaces, we are led to consider objects very like groups, but for which the multiplication is not defined everywhere. These objects are called *groupoids*, and their theory is developed in Chapter 3.

Chapter 4 contains the main properties of fundamental groups, including van Kampen's theorem which gives the fundamental group of the union of two spaces in terms of the fundamental groups of the spaces and their intersection. The proof of this theorem is simplified by the use of the fundamental groupoid. There follows a brief account of covering spaces, the deeper properties being left to a later chapter. It is shown how these properties can be used to obtain results about the complex plane; in particular, to give a proof of the Fundamental Theorem of Algebra. The chapter concludes with a discussion of some surprising situations which can occur when the spaces considered are not nice enough.

In Chapter 5 we look at graphs and complexes. Part of their theory parallels that of Chapter 4, but without the topological complications that may occur; occasionally, however, there is some cost in avoiding these technicalities. Among the material in this chapter is a discussion of the fixed subgroup of an automorphism of a free group. This uses the very easy approach of Goldstein and Turner, and its recent extension by M. Cohen and Lustig.

In Chapter 6 we discuss coverings of spaces and complexes. This is put to use in group theory in Chapter 7. In that chapter we prove Schreier's theorem on subgroups of a free group, and Kurosh's theorem on subgroups of a free product. These theorems are proved both using coverings and by purely algebraic means, and the connection between these two approaches is investigated in detail. The material in this chapter is perhaps the heart of the interplay between topology and group theory.

The long Chapter 8 is devoted to the theory of groups acting on trees, due to Bass and Serre. To some extent, this can be described as a one-dimensional rewriting of the theory of coverings, and this one-dimensional aspect makes it easier to work with than the other approaches, once the main theorems have been proved. We begin with a discussion of free actions of groups on trees, which leads to yet another proof of Schreier's theorem. We then look at Nielsen's approach to subgroups of free groups. After that we develop the general results of Bass-Serre theory. These results are then applied, in the first place to give a simple proof of the Kurosh Subgroup Theorem and related theorems about the subgroups of amalgamated free products and HNN extensions. Further applications are to Grushko's theorem on free products, and to the theorems of M. Hall and of Howson about finitely generated subgroups of free groups. These theorems find their best versions, including various generalisations, in this context. The chapter concludes with a discussion of general ways of constructing trees, including a statement of the important new theorem due to Dicks and Dunwoody, showing how this theorem can be applied to give the structure of groups with a free subgroup of finite index.

Chapter 9 is about decision problems for groups. It begins with a discussion about the nature of decision problems. Some easy decision problems are then looked at, together with an introduction to the word problem. We then show that there is a group with unsolvable word problem, and we prove Higman's Embedding Theorem, in both cases using the method of modular machines, which provides short proofs of these results. Various other related problems are then looked at, including the unsolvability of the isomorphism problem. The chapter concludes with a modern proof of Magnus's Freiheitssatz for one-relator groups and his solution of the word problem for these groups.

Chapter 10 is a quick account, without proofs, of some other topics in the theory, including small cancellation theory and Whitehead's results on automorphisms.

This book is an expanded version of a book produced in the Queen Mary College Lecture Notes series some ten years ago. The latter was an account of a two-term course of postgraduate lectures given in 1976-7.

In my original two-term course I covered most of the material in Chapters 1 to 7. For a one-semester course the best choice is to concentrate either on the group theory or the topology. I am currently giving a one-semester course, in which I cover Chapter 1, the first three sections of Chapter 5, and the part of Chapter 7 dealing with Schreier's Theorem using graphs but not coverings. Alternatively, Chapters 2,4, possibly parts of 5, 6, and some of 7 could form a course on the fundamental group, while Chapter 2 and parts of Chapter 4 (especially the early part and the section on the circle and the complex plane) could even form part of an undergraduate topology course.

Readers will notice that I have used the phrase "it is easy to see that..." very often. I believe that, whenever I have used this phrase, all the results are genuinely easy to prove. Readers should regard such results as additional exercises.

I am grateful to Ian Chiswell for reading the early parts of this book and for pointing out obscurities and inaccuracies, which I have endeavoured to correct.

The typist of a complex mathematical work like this deserves credit, and I am happy to acknowledge all my hard work in typing this book. I would also like to thank Franz Schmerbeck, the creator of the Signum word processor (running on the Atari ST), without whose valuable work the production of this book would have been much more difficult.

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