PART I

PRINCIPAL LECTURES

Pulse Reflection by a Random Medium

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1. Introduction

The study of pulse propagation in one dimensional random media arises in many applied contexts. While reflection and transmission of monochromatic waves was studied extensively some time ago [1-6 and references therein], new and perhaps surprising results emerge in the study of pulses that cannot be understood simply from the single frequency analysis by Fourier synthesis. The numerical study of Richards and Menke [7] drew our attention to these questions and led to [8] and [9]. Here we extend and simplify the analysis of [8] and give several new results. The computations are at a formal level comparable to the one in [8].

In [8] we analyzed the reflection of a pulse that is broad compared to the size of the inhomogeneities of the random medium. The random functions characterizing the medium properties were statistically homogeneous. We gave a rather complete description of the reflected signal process in a well defined asymptotic limit in which it has a canonical structure. We introduced the notion of a windowed process and showed that the canonical reflection process is windowed and Gaussian. We found a scaling law for the power spectral density but not its explicit form. All this was subjected to extensive numerical simulations in [9] where an intrinsic scaling, localization length scaling, was introduced that makes comparison to the theory much more reliable. This intrinsic scaling idea is not fully understood theoretically but seems to be very promising.

In this paper we extend the analysis to random media that are not statistically homogeneous. The incident pulse is now broad compared to the size of the inhomogeneities but short compared to the scale of variation of the mean properties. The pulse can thus resolve the mean structure while the fluctuations affect the reflected signal in a canonical way. The problem is formulated in section 2. The calculations are done in the frequency domain as in [8] and at the level of second moments (power spectra) they differ little from similar calculations in [1] for example. In section 3 we state the results which include a new equation (3.4) for the canonical power spectral density in a statistically inhomogeneous random medium. In the special statistically homogeneous case of [8] they can actually be solved explicitly (formula (3.7)). We were not able to do this in [8]. In section 4 we show how the results are obtained, including formula (3.7). Appendix A contains a brief outline of the main result in the asymptotic analysis of stochastic equations that we need here (cf also [8]).

Since all calculations here are at the level of the single (or finite) frequency results of [1] why is the analysis of pulse statistics so different? A careful look at what follows shows that we have frequently interchanged limits in the course of taking Fourier transforms and doing the small parameter asymptotics. To justify these interchanges one must do the small parameter asymptotics in an infinite dimensional setting (simultaneously for all frequencies) which is much more involved. If this seems pedantic, given that our results are correct, consider showing that the limit pulse statistics are Gaussian (this is not attempted here). In [8] we gave a finite dimensional argument for this that was incomplete and not very transparent. In the more general setting [10] the Gaussian property comes out much more naturally. It is worth noting that even though the limit law is Gaussian, the usual central limit methods do not apply because the necessary asymptotic independence (in the frequency domain) is very weak and controlled largely by the geometrical optics limit, not the mixing properties of the random medium.

2. Formulation and Scaling

We consider a one-dimensional acoustic wave propagating in a random slab of material occupying the half space $x < 0$. We will analyze in detail the backscatter at $x = 0$.

Let $p(t,x)$ be the pressure and $u(t,x)$ velocity. The linear conservation laws of momentum and mass governing acoustic wave propagation are

$$\rho(x)\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}p(t,x) = 0 \qquad (2.1)$$

$$\frac{1}{K(x)}\frac{\partial}{\partial t}p(t,x) + \frac{\partial}{\partial x}u(t,x) = 0$$

where ρ is density and K the bulk modulus. We define means of ρ and $\frac{1}{K}$ as

$$\rho_o = E\,[\rho] \tag{2.2}$$

$$\frac{1}{K_o} = E\,[\frac{1}{K}]\,.$$

In the special case that ρ and K are stationary random functions of position x, ρ_o, K_o are the constant parameters of effective medium theory. That is, a pulse of long wavelength will propagate over distances that are not too large as if in a homogeneous medium with "effective" constant parameters ρ_o, K_o, and hence with propagation speed

$$c_o = \sqrt{K_o/\rho_o}\;. \tag{2.3}$$

We consider here the case where ρ_o, K_o, c_o are not constant, but vary slowly compared to the spatial scale, l_o, of a typical inhomogeneity. We may take the "microscale" l_o to be the correlation length of ρ and $\frac{1}{K}$. We introduce a "macroscale", l_o/ε^2, where $\varepsilon > 0$ is a small parameter. It is on this macroscale that ρ_o, K_o, and other statistics of ρ and K are allowed to vary. We thus write the density and bulk modulus on the macroscale in the following scaled form.

$$\rho(x) = \rho_o\,(\frac{x}{l_o})\left[1 + \eta(\frac{x}{l_o}\,,\,\frac{x}{\varepsilon^2 l_o})\right] \tag{2.4}$$

$$\frac{1}{K(x)} = \frac{1}{K_o(x/l_o)}\left[1 + \nu(\frac{x}{l_o}\,,\,\frac{x}{\varepsilon^2 l_o})\right]$$

where the random fluctuations η and ν have mean zero and slowly varying statistics. The mean density ρ_o and the mean bulk modulus K_o are assumed to be differentiable functions of x.

Equations (2.1) are to be supplemented with boundary conditions at $x = 0$ corresponding to different ways in which the pulse is generated at the interface. In the cases analyzed below the pulse width is assumed to be on a scale intermediate between the microscale and the macroscale. That is, the pulse is broad compared to the size of the random inhomogeneities, but short compared to the non-random variations. Thus the small scale structure will introduce only random effects which the pulse is too broad to probe in detail. In contrast, the pulse is chosen to probe the non-random macroscale, from which it reflects and refracts in the manner of ray theory (geometrial optics). We will recover macroscopic variations of the medium by examination of reflections at $x = 0$.

Let typical values of ρ_o, K_o be $\bar{\rho}$, $\bar{K}$ with $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$. Then for $f(t)$ a smooth function of compact support in $[0,\infty)$ we define the incident pulse by

$$f^{\varepsilon}(t) = \frac{1}{\varepsilon^{1/2}} f\left(\frac{\bar{c}\, t}{\varepsilon\, l_o}\right). \tag{2.5}$$

This pulse, f^{ε}, will be convolved with the appropriate Green's function depending on how the wave is excited at the interface. The pre-factor $\varepsilon^{-1/2}$ is introduced to make the energy of the pulse independent of the small parameter ε.

We consider here the "matched medium" boundary condition. It is assumed that the wave is incident on the random medium occupying $x < 0$ from a homogeneous medium occupying $x > 0$ and characterized by the constant parameters $\rho_o(0)$, $K_o(0)$. One may similarly consider an unmatched medium where ρ_o, K_o are discontinuous at $x = 0$, but we do not carry this out here. To obtain the Green's function for this problem we introduce the initial-boundary condition for a left-travelling wave which strikes $x = 0$ a time $t = 0$

$$u = l_o\, \delta\left(t + \frac{x}{c_o(0)}\right) \tag{2.6}$$

$$p = -\, l_o\, \rho_o(0)\, c_o(0)\, \delta\left(t + \frac{x}{c_o(0)}\right)$$

The Green's function G will then be a right-going wave in $x > 0$ and as $x \downarrow 0$

$$G = \frac{1}{2}\left[u(t,0) - \frac{p(t,0)}{(\rho_o(0) c_o(0))}\right] \tag{2.7}$$

We non-dimensionalize by setting

$$x' = x / l_o \quad p' = p / \bar{\rho}\, \bar{c}^{\,2} \tag{2.8}$$

$$t' = \bar{c} t / l_o \quad u' = u / \bar{c}$$

By inserting (2.8) into the above equations, and dropping primes, it can be shown that without loss of generality $\bar{K}$, $\bar{\rho}$, $\bar{c}$, l_o may be taken equal to unity, after K, ρ, c are replaced by their normalized forms.

We will determine the statistics of the Green's function convolved with the pulse f^{ε}. Let

$$G^{\varepsilon}_{t,f}(\sigma) = (G * f^{\varepsilon})(t + \varepsilon\sigma) \tag{2.9}$$

$$= \int_0^{t+\varepsilon\sigma} G(t + \varepsilon\sigma - s)\, f^{\varepsilon}(s)\, ds\, .$$

We consider the above expression as a stochastic process in σ, with t held fixed. That is, for each t we consider a "time window" centered at t, and of duration on the order of a pulse width, with the parameter σ measuring time within this window.

For the analysis of this problem, we Fourier transform in time, choosing a frequency scale appropriate to the pulse $f^{\varepsilon}(t)$. Thus, letting

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt \tag{2.10}$$

we transform (2.1) by

$$\hat{u}(\omega, x) = \int e^{i\omega t/\varepsilon} u(t,x)\, dt \tag{2.11}$$

$$\hat{p}(\omega, x) = \int e^{i\omega t/\varepsilon} p(t,x)\, dt$$

so that

$$G^{\varepsilon}_{t,f}(\sigma) = \frac{1}{2\pi\varepsilon^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega[t+\varepsilon\sigma]/\varepsilon} \hat{f}(\omega)\, \hat{G}(\omega) d\omega\, . \tag{2.12}$$

In (2.12) $\hat{G}$ is the appropriate combination of $\hat{u}$, $\hat{\rho}$ obtained by Fourier transform of (2.7).

From (2.1), (2.4), (2.11), $\hat{u}$, $\hat{p}$ satisfy

$$\frac{\partial}{\partial x}\hat{p} = \frac{i\omega}{\varepsilon} \rho_o(x)\, [1 + \eta(x, \frac{x}{\varepsilon^2})]\, \hat{u} \tag{2.13}$$

$$\frac{\partial}{\partial x}\hat{u} = \frac{i\omega}{\varepsilon} \frac{1}{K_o(x)} [1 + \nu(x, \frac{x}{\varepsilon^2})]\, \hat{p}\, .$$

In the frequency domain a radiation condition as $x \to -\infty$, is required for (2.13). One way to do this is to terminate the random slab at a finite point $x = -L$, and assume the medium is not random for $x > -L$. We can later let $L \to -\infty$ but in any case the reflected signal up to a time t is not affected by how we terminate the slab at a sufficiently distant point $-L$. This is a consequence of the hyperbolicity of (2.1).

We next introduce a right going wave A and a left going wave B, with respect to the macroscopic medium. Let the travel time in the macroscopic medium be given by

$$\tau(x) = \int_x^0 \frac{ds}{c_o(s)}\ , \quad x < 0 \tag{2.14}$$

We define A, B by

$$\hat{u} = \frac{1}{(K_o\rho_o)^{1/4}} [A\, e^{-i\omega\tau/\varepsilon} + B\, e^{i\omega\tau/\varepsilon}]$$

$$\hat{p} = (K_o\, \rho_o)^{1/4}\, [\, A\, e^{-i\,\omega\tau/\varepsilon} - B\, e^{i\,\omega\tau/\varepsilon}\,] \tag{2.15}$$

Putting (2.14), (2.15) into (2.13) yields equations for A, B. Define the random functions $m^\varepsilon(x)$ and $n^\varepsilon(x)$ by

$$m^\varepsilon(x) = m(x,\ x/\varepsilon^2) = \frac{1}{2}\,[\eta(x,x/\varepsilon^2) + \nu(x,x/\varepsilon^2)] \tag{2.16}$$

$$n^\varepsilon(x) = n(x,\ x/\varepsilon^2) = \frac{1}{2}\,[\eta(x,x/\varepsilon^2) - \nu(x,x/\varepsilon^2)]$$

Then

$$\frac{d}{dx}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{i\,\omega}{\varepsilon}\left(\frac{\rho_o}{K_o}\right)^{1/2}\begin{bmatrix} m^\varepsilon & n^\varepsilon\, e^{2i\omega\tau/\varepsilon} \\ -n^\varepsilon\, e^{-2i\,\omega\tau/\varepsilon} & -m^\varepsilon \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$+\frac{1}{4}\,\frac{(K_o\,\rho_o)'}{(K_o\,\rho_o)}\begin{bmatrix} 0 & e^{2i\omega\tau/\varepsilon} \\ e^{-2i\omega\tau/\varepsilon} & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{2.17}$$

We take as boundary conditions for (2.17) that there is no right-going wave at $x = -L$, and that there is a unit left-going wave at $x = 0$.

$$A(-L) = 0 \qquad B(0) = 1 \tag{2.18}$$

$$B(-L) = T \qquad A(0) = R = R^\varepsilon(-L,\ \omega)$$

Here T is the transmission coefficient for the slab, and $R^\varepsilon(-L,\ \omega)$ is the reflection coefficient. From (2.6), (2.7) we see that

$$\hat{G} = R^\varepsilon\,(-L,\ \omega)\,. \tag{2.19}$$

We introduce the fundamental matrix solution of the linear system (2.17). That is, let $Y(x,\ -L)$ satisfy (2.17) with the initial condition that $Y(-L,\ -L) = I$ the 2 x 2 identity. From symmetries in (2.17) it is apparent that if $(a,\bar{b})^T$ is a vector solution (bar denotes complex conjugate and T transpose), then so is $(b,\ \bar{a})^T$. Thus

$$Y = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}. \tag{2.20}$$

Furthermore, since the system has trace zero, Y has determinant one. Hence

$$|\,a\,|^2 - |\,b\,|^2 = 1\,. \tag{2.21}$$

Now the reflection coefficient R may be expressed in terms of a, b, by writing (2.18) in terms of propagators, i.e.

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} 0 \\ T \end{bmatrix} = \begin{bmatrix} R \\ 1 \end{bmatrix}$$

and hence

$$R = \frac{b}{\bar{a}}, \qquad T = \frac{1}{\bar{a}} \tag{2.22}$$

Now from (2.17), (2.20) we have that

$$\frac{da}{dx} = \frac{i\,\omega}{\varepsilon}\left[\frac{\rho_o}{K_o}\right]^{1/2}\left[m^\varepsilon\, a + n^\varepsilon\, \bar{b}\, e^{2i\,\omega\tau/\varepsilon}\right] + \frac{1}{4}\,\frac{(\rho_o K_o)'}{\rho_o K_o}\,\bar{b}\, e^{2i\omega\tau/\varepsilon}$$

$$\frac{d\,\bar{b}}{dx} = -\frac{i\,\omega}{\varepsilon}\left[\frac{\rho_o}{K_o}\right]^{1/2}\left[n^\varepsilon\, a\, e^{-2i\omega\tau/\varepsilon} + m^\varepsilon \bar{b}\right] + \frac{1}{4}\,\frac{(\rho_o K_o)'}{\rho_o K_o}\, a\, e^{-2i\omega\tau/\varepsilon} \tag{2.23}$$

$$a\,(-L) = 1,\ b(-L) = 0\,.$$

Therefore, from (2.22), (2.23) we can derive the Riccati equation for R

$$\frac{dR^\varepsilon}{dx} = \frac{i\,\omega}{\varepsilon}\left[\frac{\rho_o}{K_o}\right]^{1/2}\left[n^\varepsilon\, e^{\,2i\,\omega\tau/\varepsilon} + 2m^\varepsilon\, R^\varepsilon + n^\varepsilon (R^\varepsilon)^2\, e^{-2i\omega\tau/\varepsilon}\right]$$

$$+\frac{1}{4}\,\frac{(\rho_o K_o)'}{(\rho_o K_o)}\left[e^{2i\,\omega\tau/\varepsilon} - (R^\varepsilon)^2\, e^{-2i\,\omega\tau/\varepsilon}\right] \tag{2.24}$$

$$R^\varepsilon(-L) = 0 \quad .$$

The boundary condition at $-L$ in (2.24) is for termination of the random slab by a uniform medium. If the medium is homogeneously random beyond $-L$ ($\rho_o(x)$, $K_o(x)$ constant) then we will have total reflection at $-L$ because the wave cannot penetrate the random medium to infinite depth. In fact in a statistically homogeneous random medium we have that

$$|\,T\,| \to 0 \ \text{ as } \ L \to -\infty \tag{2.25}$$

exponentially fast which follows from Furstenberg's theorem [11,12]. Since (2.21), (2.22) imply that $|\,R\,|^2 + |\,T\,|^2 = 1$ we have

$$|\,R\,| \to 1 \text{ as } \ L \to -\infty\,. \tag{2.26}$$

It is convenient to analyze (2.24) with a **totally reflecting termination**, so that

$$R^\varepsilon = e^{-i\,\psi^\varepsilon}\,. \tag{2.27}$$

and the number of degrees of freedom is reduced by one. This simplification, not possible when we do have transmission, was not made in [8]. Putting (2.27) into (2.24) yields

$$\frac{d}{dx}\psi^{\varepsilon} = -\frac{\omega}{\varepsilon}\left[\frac{\rho_o(x)}{K_o(x)}\right]^{1/2}\left[2\,m^{\varepsilon}(x) + 2n^{\varepsilon}(x)\cos\left(\psi^{\varepsilon} + \frac{2\omega\tau(x)}{\varepsilon}\right)\right] \tag{2.28}$$

$$+\frac{1}{2}\frac{(\rho_o K_o)'}{\rho_o K_o}\sin\left(\psi^{\varepsilon} + \frac{2\omega\tau(x)}{\varepsilon}\right)$$

and we take ψ^{ε} to be asymptotically stationary as $x \to -\infty$.

To recapitulate, the asymptotically stationary solution of (2.28), evaluated at x=0 is put into (2.27) to yield the totally reflecting reflection coefficient R^{ε} at frequncy ω. The frequency domain Green's function is then given by (2.19) The result is then transformed back to the time domain by (2.12).

3. Statement of the main results

Let $G^{\varepsilon}_{t,f}(\sigma)$ be the reflection process observed at $x=0$ within the time window centered at t. Then $G^{\varepsilon}_{t,f}(\cdot)$ converges weakly as $\varepsilon \downarrow 0$ to a stationary Gaussian process with mean zero and power spectral density

$$S_t(\omega) = |\,\hat{f}(\omega)\,|^2\,\mu(t,\omega)\,, \tag{3.1}$$

The normalized power spectral density μ is computed as follows.

Let α_{nn} be the integral of the second moment of the medium properties defined by

$$\alpha_{nn}(x) = \int_0^{\infty} E\,[n(x,y)\,n(x,y+s)]\,ds\ . \tag{3.2}$$

Let $\tau(x)$ be travel time to depth x defined by (2.14), and let $\bar{x}(\tau)$ be its inverse which is depth reached up to time t in the medium without fluctuations. Define

$$\gamma(\tau) = \frac{\alpha_{nn}(\bar{x}(\tau))}{c_o(\bar{x}(\tau))}\ . \tag{3.3}$$

Let $W^{(N)}(\tau,\,t,\,\omega)$, $N = 0,1,2...$ be the solution of

$$\frac{\partial W^{(N)}}{\partial\tau} + 2N\frac{\partial W^{(N)}}{\partial t} - 2\omega^2\,\gamma(\tau)\Big\{[N+1]^2\,W^{(N+1)}$$

$$-2N^2W^{(N)} + [N-1]^2W^{(N-1)}\Big\} = 0 \tag{3.4}$$

for

$$t,\ \tau > 0\,, \quad N = 0,1,2,\cdots$$

with

$$W^{(N)} \equiv 0 \ \text{for} \quad t < 0\,,\ N < 0\,.$$

and

$$W^{(N)}\,(0,t,\omega) = \delta(t)\,\delta_{N,1}\,. \tag{3.5}$$

Then

$$\mu(t,\omega) = \lim_{\tau\to\infty} W^{(0)}\,(\tau,t,\omega)\,. \tag{3.6}$$

The system (3.4) is hyperbolic so it is not necessary to take a limit in (3.6) because $W^{(0)}$ is constant for $\tau > t/2$. Thus

$$\mu(t,\omega)=W^{(0)}(\frac{t}{2},t,\omega) \tag{3.6a}$$

For the case of a homogeneous medium $[c_o,\ \gamma = const = \tilde{\gamma}]$ the normalized power spectral density can be computed explicitly.

$$(BCI) \quad \mu_1(t,\omega) = \frac{\omega^2\tilde{\gamma}}{[1+\omega^2\tilde{\gamma}t\,]^2} \tag{3.7}$$

4. Calculation of Power Spectral Density.

We next calculate the power spectrum, as $\varepsilon \downarrow 0$, or the reflection process $G^{\varepsilon}_{t,f}\,(\sigma)$. From (2.9) we have the correlation function $C^{\varepsilon}_{t,f}$

$$C^{\varepsilon}_{t,f}\,\sigma) \equiv E[G^{\varepsilon}_{t,f}(\sigma)\,G^{\varepsilon}_{t,f}(0)] \tag{4.1}$$

$$= \frac{1}{4\,\pi^2\,\varepsilon}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, e^{-i\,\omega_1 t/\varepsilon}\, e^{-i\omega,\sigma}\, e^{i\omega_2\, t/\varepsilon}$$

$$\cdot\,\hat{f}\,(\omega_1)\,\bar{\hat{f}}\,(\omega_2)\,E[\hat{G}\,(\omega_1)\,\bar{\hat{G}}(\omega_2)]\,.$$

Let