CHAPTER I THE ALGEBRA OF GROUPOIDS

In this chapter we define groupoids and their morphisms and give the basic algebraic definitions and constructions of subgroupoids, quotient groupoids, kernels of morphisms, products of groupoids and other standard concepts. We do not address the algebraic theory of groupoids for its own sake, and we do not prove any of the deeper results from the algebraic theory.

An interesting algebraic theory of groupoids exists, and was begun by Brandt and by Baer in the 1920's, well before Ehresmann made the concept of groupoid central to his vision of differential geometry. However the algebraic theory is primarily concerned with problems which are largely trivial for categories of transitive groupoids and there is therefore no reason for us to treat it here. See Higgins (1971) for a full account and further references, and Brown (1968) for an account which is more accessible to the non-algebraist, though less comprehensive than Higgins'. Much material on the algebraic theory of groupoids, from a different point of view to that of the work cited above, can be extracted from Ehresmann (1965). See also Clifford and Preston (1961, §3.3).

The examples given in this chapter are examples of topological or differentiable groupoids, presented without their topological, or smooth, structures. We have managed to avoid giving examples which can arise only in the purely algebraic setting.

The development of the algebraic theory of groupoids has been succinctly chronicled by Higgins (1971, pp. 171-172). The examples, as has been said, belong to the topological and differentiable theories, and will be sourced when they reappear in full in later chapters.

§1. Groupoids

A groupoid is a complicated structure and we will spend a little time in giving a full definition and, in the process, introduce the notation to be used in these notes.

Groups commonly arise as the structures natural to sets of automorphisms of mathematical objects. In differential geometry one frequently encounters families of mutually isomorphic objects, the basic example being the set of tangent spaces to a manifold, and the way in which the members of such a family relate to each other is captured by taking as the set of 'automorphisms', not merely the automorphisms of each individual object, but all isomorphisms between each pair of objects in the family. The resulting system of isomorphisms has the structure of a groupoid. Of course, like groups, groupoids also often arise in other ways, not related to automorphisms.

To illustrate the concept of groupoid, we take as example the set, denoted $\Pi(TB)$, of all linear isomorphisms between the various tangent spaces to a manifold B. Each such isomorphism $\xi\colon T(B)_x \to T(B)_y$ has associated with it two points of B, namely the points x and y which label the tangent spaces which are its domain and range; we denote x by $\alpha(\xi)$ and y by $\beta(\xi)$ and call $\alpha,\beta\colon \Pi(TB) \to B$ the source and target projections of $\Pi(TB)$; the isomorphism ξ can be composed with an isomorphism $\eta\colon T(B)_{y'} \to T(B)_z$ iff $y' = y$, that is, iff $\alpha(\eta) = \beta(\xi)$. Thus composition is a partial multiplication on $\Pi(TB)$ with domain the set $\Pi(TB)*\Pi(TB) = \{(\eta,\xi) \in \Pi(TB) \times \Pi(TB) \mid \alpha(\eta) = \beta(\xi)\}$. Note that when the composition $\eta\xi$ is defined, we have $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$. This partial multiplication has properties which resemble the properties of a group multiplication as closely as is possible: each point $x \in B$ has associated with it the identity isomorphism $id_{T(B)_x}$, here denoted $\tilde{x}$, and the elements $\tilde{x}, x \in B$, act as unities for every multiplication in which they can take part; each isomorphism $\xi\colon T(B)_x \to T(B)_y$ has an inverse isomorphism $\xi^{-1}\colon T(B)_y \to T(B)_x$ and $\xi\xi^{-1}$ and $\xi^{-1}\xi$ are the unities $\widetilde{\beta(\xi)}$ and $\widetilde{\alpha(\xi)}$ respectively. These properties are abstracted into the following definition.

<u>Definition 1.1.</u> A <u>groupoid</u> consists of two sets Ω and B, called respectively the <u>groupoid</u> and the <u>base</u>, together with two maps α and β from Ω to B, called respectively the <u>source</u> and <u>target projections</u>, a map $\varepsilon\colon x \mapsto \tilde{x}$, $B \to \Omega$ called the <u>object inclusion map</u>, and a partial multiplication $(\eta,\xi) \mapsto \eta\xi$ in Ω defined on the set $\Omega*\Omega = \{(\eta,\varepsilon) \in \Omega \times \Omega \mid \alpha(\eta) = \beta(\xi)\}$, all subject to the following conditions:

(i) $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$ for all $(\eta,\xi) \in \Omega*\Omega$;

(ii) $\zeta(\eta\xi) = (\zeta\eta)\xi$ for all $\zeta,\eta,\xi \in \Omega$ such that $\alpha(\zeta) = \beta(\eta)$ and $\alpha(\eta) = \beta(\xi)$;

(iii) $\alpha(\tilde{x}) = \beta(\tilde{x}) = x$ for all $x \in B$;

(iv) $\xi\widetilde{\alpha(\xi)} = \xi$ and $\widetilde{\beta(\xi)}\xi = \xi$ for all $\xi \in \Omega$;

(v) each $\xi \in \Omega$ has a (two-sided) inverse ξ^{-1} such that $\alpha(\xi^{-1}) = \beta(\xi)$, $\beta(\xi^{-1}) = \alpha(\xi)$ and $\xi^{-1}\xi = \widetilde{\alpha(\xi)}$, $\xi\xi^{-1} = \widetilde{\beta(\xi)}$. //

Elements of B may be called <u>objects</u> of the groupoid Ω and elements of Ω may be called <u>arrows</u>. The arrow $\tilde{x}$ corresponding to an object $x \in B$ may also be called the <u>unity</u> or <u>identity</u> corresponding to x. To justify this terminology and to prove that the inverse in (v) is unique, we have the following proposition.

Proposition 1.2. Let Ω be a groupoid with base B, and consider $\xi \in \Omega$ with $\alpha(\xi) = x$ and $\beta(\xi) = y$.

(i) If $\eta \in \Omega$ has $\alpha(\eta) = y$ and $\eta\xi = \xi$, then $\eta = \tilde{y}$.
If $\zeta \in \Omega$ has $\beta(\zeta) = x$ and $\xi\zeta = \xi$, then $\zeta = \tilde{x}$.

(ii) If $\eta \in \Omega$ has $\alpha(\eta) = y$ and $\eta\xi = \tilde{x}$, then $\eta = \xi^{-1}$.
If $\zeta \in \Omega$ has $\beta(\zeta) = x$ and $\xi\zeta = \tilde{y}$, then $\zeta = \xi^{-1}$.

Proof. Exercise. //

In place of the phrase "a groupoid with base B", we will often write "a groupoid on B". For a groupoid Ω on B and $x,y \in B$ we will write Ω_x for $\alpha^{-1}(x)$, Ω^y for $\beta^{-1}(y)$ and Ω_x^y for $\Omega_x \cap \Omega^y$. To avoid cumbersome suffices we will sometimes denote "$\xi \in \Omega_x^y$" by "$\xi\colon x \to y$". The set Ω_x is the <u>α-fibre</u> over x and Ω^y is the <u>β-fibre</u> over y. The set Ω_x^x, obviously a group under the restriction of the partial multiplication in Ω, is called the <u>vertex group</u> at x. Some writers call Ω_x^x the <u>isotropy group</u> at x. For any subsets $U,V \subseteq B$ we likewise write Ω_U, Ω^V and Ω_U^V for $\alpha^{-1}(U)$, $\beta^{-1}(V)$ and $\Omega_U \cap \Omega^V$, respectively.

Many authors denote Ω_x^y by $\Omega(x,y)$, call Ω_x the <u>star</u> of Ω at x and denote it by $St_\Omega x$, and call Ω^y the <u>co-star</u> of Ω at y and denote it by $Cost_\Omega y$.

The following examples are of basic importance.

Example 1.3. Any set B may be regarded as a groupoid on itself with $\alpha = \beta = id_B$ and every element a unity. Groupoids in which every element is a unity have been given a variety of names; we will call them <u>base groupoids</u>. //

Example 1.4. Let B be a set and G a group. We give $B \times G \times B$ the structure of a groupoid on B in the following way: α is the projection onto the third factor of $B \times G \times B$ and β is the projection onto the first factor; the object inclusion map is $x \mapsto \tilde{x} = (x,1,x)$ and the partial multiplication is $(z,h,y')(y,g,x) = (z,hg,x)$, defined iff $y' = y$. The inverse of (y,g,x) is (x,g^{-1},y). This is called the <u>trivial groupoid</u> on B with group G.

In particular, any group may be considered to be a groupoid on any singleton set, and any cartesian square $B \times B$ is a groupoid on B. //

Example 1.5. Let X be an equivalence relation on a set B. Then $X \subseteq B \times B$ is a groupoid on B with respect to the restriction of the structure defined in 1.4. Each α-fibre X_x, $x \in B$, may be naturally identified with the equivalence class containing x.

Groupoids Ω such as this, in which each Ω_x^y is either empty or singleton, are sometimes called principal groupoids (see Renault (1980)). We use this term with a different meaning (see II 2.9). //

Example 1.6. Let $G \times B \to B$ be an action of a group G on a set B. Give $G \times B$ the structure of a groupoid on B in the following way: α is the projection onto the second factor of $G \times B$ and β is the action $G \times B \to B$ itself; the object inclusion map is $x \mapsto \tilde{x} = (x,1)$ and the partial multiplication is $(g_2,y)(g_1,x) = (g_2g_1,x)$, defined iff $y = g_1x$. The inverse of (g,x) is (g^{-1},gx). We propose to call $G \times B$ the action groupoid of $G \times B \to B$.

The α-fibre $(G \times B)_x$ is $G \times \{x\}$, and the β-fibre can also be identified with the group G. The vertex group $(G \times B)_x^x$ is naturally isomorphic to the isotropy group G_x.

This construction can be generalized. See II 4.20. //

Example 1.7. Applying the construction of 1.6 to the action $\mathbf{R} \times S^1 \to S^1$, $(t,z) \mapsto e^{2\pi it}z$ gives a groupoid structure on the cylinder $\mathbf{R} \times S^1$. The base may be identified with the circle $t = 0$, the α-fibres are straight lines orthogonal to $t = 0$, the β-fibres are the helices which make an angle of 45° with the circles t = constant, and the vertex groups are the $\mathbf{Z} \times \{z\}$ for $z \in S^1$.

The reader may construct similarly visualizable examples on the torus, using the actions $S^1 \times S^1 \to S^1$, $(w,z) \mapsto w^nz$, for given $n \in \mathbf{Z}$. However no truly typical example of a groupoid of the type with which we shall be concerned in later chapters can be visualized by means of an embedding in $\mathbf{R}^3$. //

Example 1.8. Let B be a topological space. Then the set $\pi(B)$ of homotopy classes $\langle c \rangle$ rel endpoints of paths $c: [0,1] \to B$ is a groupoid on B with respect to the following structure: the source and target projections are $\alpha(\langle c \rangle) = c(0)$ and $\beta(\langle c \rangle) = c(1)$, the object inclusion map is $x \mapsto \tilde{x} = \langle \kappa_x \rangle$, where κ_x is the path constant at x, and the partial multiplication is $\langle c' \rangle \langle c \rangle = \langle c'c \rangle$ where $c'c$ is the standard concatenation of c followed by c', namely $(c'c)(t) = c(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$, $(c'c)(t) = c'(2t-1)$ for $\frac{1}{2} \leqslant t \leqslant 1$. The inverse of $\langle c \rangle$ is $\langle c^{\leftarrow} \rangle$ where $c^{\leftarrow}$ is the reverse of the path c, namely $c^{\leftarrow}(t) = c(1-t)$.

Note that many authors take c'c to be c' followed by c, defined iff $c'(1) = c(0)$. The groupoid $\mathcal{T}(B)$ may also be defined using paths of variable length; for this see, for example, Brown (1968).

$\mathcal{T}(B)$ is the <u>fundamental groupoid</u> of B; its vertex groups are the fundamental groups $\pi_1(B,x)$, $x \in B$, and if B is path-connected, locally path-connected and semi-locally simply connected, then its α-fibres are the sets underlying the universal covering spaces of B.

There are now a number of beginning texts on algebraic topology which introduce the concept of fundamental group via that of the fundamental groupoid, but most make little use of the groupoid structure. The first account of elementary homotopy theory to make effective use of the algebraic structure of $\mathcal{T}(B)$ was Brown (1968). //

<u>Example 1.9</u>. Let $p: M \to B$ be a surjective map. Let $\Pi(M)$ denote the set of all bijections $\xi: M_x \to M_y$ for $x,y \in B$, where $M_x = p^{-1}(x)$, $x \in B$. Then $\Pi(M)$ is a groupoid on B with respect to the following structure: for $\xi: M_x \to M_y$, $\alpha(\xi)$ is x and $\beta(\xi)$ is y; the object inclusion map is $x \mapsto \tilde{x} = \mathrm{id}_{M_x}$, and the partial multiplication is the composition of maps. The inverse of $\xi \in \Pi(M)$ is its inverse as a map. $\Pi(M)$ is called the <u>frame groupoid</u> of (M,p,B).

Many variants of this fundamental example will be given in later chapters. //

<u>Example 1.10</u>. Let $P(B,G,\pi)$ be a principal bundle. Let G act on $P \times P$ to the right by $(u_2,u_1)g = (u_2g,u_1g)$; denote the orbit of (u_2,u_1) by $\langle u_2,u_1\rangle$ and the set of orbits by $\frac{P \times P}{G}$. Then $\frac{P \times P}{G}$ is a groupoid on B with respect to the following structure: the source and target projections are $\alpha(\langle u_2,u_1\rangle) = \pi(u_1)$, $\beta(\langle u_2,u_1\rangle) = \pi(u_2)$; the object inclusion map is $x \mapsto \tilde{x} = \langle u,u\rangle$, where u is any element of $\pi^{-1}(x)$; and the partial multiplication is defined by

$$\langle u_3,u_2'\rangle\langle u_2,u_1\rangle = \langle u_3,u_1\delta(u_2',u_2)\rangle.$$

Here $\delta: P \underset{\pi}{\times} P \to G$ is the map $(ug,u) \mapsto g$ (see A§1). The condition $\alpha(\langle u_3,u_2'\rangle) = \beta(\langle u_2,u_1\rangle)$ ensures that $(u_2',u_2) \in P \underset{\pi}{\times} P$. Note that one can always choose representatives so that $u_2' = u_2$ and the multiplication is then simply

$$\langle u_3,u_2\rangle\langle u_2,u_1\rangle = \langle u_3,u_1\rangle.$$

The inverse of $\langle u_2,u_1\rangle$ is $\langle u_1,u_2\rangle$. $\frac{P \times P}{G}$ is called the groupoid associated to $P(B,G,\pi)$. //

Example 1.11. Applying 1.10 to the principal bundle $SU(2)(SO(3),\mathbf{Z}_2,\pi)$ yields a groupoid which, though it has dimension 6, is perhaps somewhat visually accessible. Here the action of $\mathbf{Z}_2 = \{I,-I\} \subseteq SU(2)$ on $SU(2)$ is by matrix multiplication and π is essentially the adjoint representation (see, for example, Miller (1972, p. 224)). The groupoid $\frac{SU(2) \times SU(2)}{\mathbf{Z}_2}$ can be naturally identified with $SO(4)$: identifying $SU(2)$ with the unit sphere in the space of quaternions $\mathbf{H}$, each pair $(p,q) \in SU(2) \times SU(2)$ defines a map $\mathbf{H} \to \mathbf{H}$, $x \mapsto pxq^{-1}$ which, as a map $\mathbf{R}^4 \to \mathbf{R}^4$, is a proper rotation. It is well-known that this map $SU(2) \times SU(2) \to SO(4)$ is an epimorphism of Lie groups with kernel $\{(I,I), (-I,-I)\}$ (see, for example, Greub (1967, p. 329)).

Thus we obtain a groupoid structure on $SO(4)$ with base $\mathbf{RP}^3$, α- and β-fibres which are 3-spheres, and vertex groups which are $\mathbf{Z}_2$'s. However it seems that the groupoid multiplication has no clear geometrical significance. //

We shall return to examples 1.4 to 1.10 in chapters II and III.

§2. Morphisms, subgroupoids and quotient groupoids

We treat the concepts listed in the title, and the related concepts of kernel, normal subgroupoid, etc., and consider the factorization of a morphism into a quotient projection, an isomorphism, and an inclusion. This factorization, fundamental in the category of groups, is valid only for certain classes of groupoid morphism, for example, those morphisms which are both piecewise-surjective and base-surjective, and those which are base-injective. In particular, base-preserving morphisms can be so factored. We mention in passing two factorizations of an arbitrary groupoid morphism into a base-preserving morphism and a morphism of another specified type.

The examples given here are tailored to those of later chapters but otherwise the material of this section comes from Higgins (1971) and Brown (1968).

Definition 2.1. Let Ω and Ω' be groupoids on B and B' respectively. A *morphism* $\Omega \to \Omega'$ is a pair of maps $\phi: \Omega \to \Omega'$, $\phi_o: B \to B'$ such that $\alpha'\circ\phi = \phi_o\circ\alpha$, $\beta'\circ\phi = \phi_o\circ\beta$ and $\phi(\eta\xi) = \phi(\eta)\phi(\xi)$, $\forall(\eta,\xi) \in \Omega*\Omega$. We also say that ϕ is a *morphism over* ϕ_o. If $B = B'$ and $\phi_o = id_B$ we say that ϕ is a *morphism over* B, or that ϕ is a *base-preserving morphism*. //

Note that the conditions $\alpha'\circ\phi = \phi_o\circ\alpha$, $\beta'\circ\phi = \phi_o\circ\beta$ ensure that $\phi(\eta)\phi(\xi)$ is defined whenever $\eta\xi$ is. Morphisms preserve unities and inverses:

Proposition 2.2. Let $\phi: \Omega \to \Omega'$, $\phi_o: B \to B'$ be a groupoid morphism. Then

(i) $\phi(\tilde{x}) = \widetilde{\phi_o(x)}$ $\quad \forall\, x \in B$,

(ii) $\phi(\xi^{-1}) = \phi(\xi)^{-1}$ $\quad \forall\, \xi \in \Omega$.

Proof. Exercise. //

For $x,y \in B$ we denote the restrictions of ϕ to $\Omega_x \to \Omega'_{\phi_o(x)}$, $\Omega^y \to \Omega'^{\phi_o(y)}$ and $\Omega^y_x \to \Omega'^{\phi_o(y)}_{\phi_o(x)}$ by ϕ_x, ϕ^y and ϕ^y_x, respectively.

Definition 2.3. A groupoid morphism $\phi: \Omega \to \Omega'$ over $\phi_o: B \to B'$ is *piecewise-surjective* (respectively, *piecewise-injective*, *piecewise-bijective*) if $\phi^y_x: \Omega^y_x \to \Omega'^{\phi_o(y)}_{\phi_o(x)}$ is surjective (respectively, injective, bijective) $\forall\, x,y \in B$.

ϕ is *base-surjective* (respectively, *base-injective*, *base-bijective*) if $\phi_o: B \to B'$ is surjective (respectively, injective, bijective).

ϕ is an *isomorphism* if $\phi: \Omega \to \Omega'$ (and hence $\phi_o: B \to B'$) is bijective. //

We will not use the words 'epimorphism' or 'monomorphism' in the algebraic context.

It is trivial to prove that a surjective (injective) morphism is base-surjective (base-injective); further, a morphism is injective iff it is base-injective and piecewise-injective, and a morphism which is base-surjective and piecewise-surjective is itself surjective. All these results are easy to prove. A surjective morphism need not be piecewise-surjective; see example 2.8 below.

Definition 2.4. Let Ω be a groupoid on B. A subgroupoid of Ω is a pair of subsets $\Omega' \subseteq \Omega$, $B' \subseteq B$, such that $\alpha(\Omega') \subseteq B'$, $\beta(\Omega') \subseteq B'$, $\tilde{x} \in \Omega'$ $\forall x \in B'$, and Ω' is closed under the partial multiplication and inversion in Ω. A subgroupoid Ω', B' of Ω, B is wide if $B' = B$ and is full if $\Omega'^y_x = \Omega^y_x$ $\forall x,y \in B'$.

The base subgroupoid or identity subgroupoid of Ω is the subgroupoid $\tilde{B} = \{\tilde{x} | x \in B\}$. The inner subgroupoid of Ω is the subgroupoid $G\Omega = \bigcup_{x \in B} \Omega^x_x$. //

A morphism of groups may be factored into a surjective morphism (the projection of the domain group onto its quotient over the kernel of the given morphism), followed by an isomorphism, followed by an injective morphism (the inclusion of the image of the given morphism into its range). For groupoid morphisms the situation is more complicated. Firstly, the image of a groupoid morphism need not be a subgroupoid; it may happen that a product $\phi(\eta)\phi(\xi)$ is defined but the product $\eta\xi$ is not and that another pair η_1, ξ_1 with $\phi(\eta_1) = \phi(\eta)$, $\phi(\xi_1) = \phi(\xi)$ and $\eta_1\xi_1$ defined cannot be found. This can occur even for morphisms of trivial groupoids: Let B be an interval on the real line, bounded away from infinity and zero, and G' the multiplicative group of positive reals and consider $B \times B \to G'$, $(y,x) \mapsto yx^{-1}$. (It is easy to prove, however, that the image of a base-injective morphism is a subgroupoid.)

Secondly, the concept of kernel for groupoid morphisms does not adequately measure injectivity. To demonstrate this failure and its consequences and describe what factorizations are possible will occupy us until 2.13.

Definition 2.5. Let Ω be a groupoid on B. A normal subgroupoid of Ω is a wide subgroupoid Φ such that for any $\lambda \in G\Phi$ and any $\xi \in \Omega$ with $\alpha\xi = \alpha\lambda = \beta\lambda$, we have $\xi\lambda\xi^{-1} \in \Phi$. //

Note that whether or not a subgroupoid is normal depends only on those of its elements which lie in its inner subgroupoid.

Definition 2.6. Let $\phi: \Omega \to \Omega'$, $\phi_o: B \to B'$ be a morphism of groupoids. Then the kernel of ϕ is the set $\{\xi \in \Omega \mid \phi(\xi) = \tilde{x}, \exists x \in B'\}$. //

Clearly the kernel of a morphism is a normal subgroupoid. The following construction of quotient groupoids shows that every normal subgroupoid is the kernel of a morphism.

Proposition 2.7. Let Φ be a normal subgroupoid of a groupoid Ω on B. Define an equivalence relation $\sim$ on B by $x \sim y \Leftrightarrow \exists\, \zeta \in \Phi$: $\alpha\zeta = x$, $\beta\zeta = y$, and denote the equivalence classes by $[x], x \in B$, and the set of equivalence classes by B/Φ. Define a second equivalence relation, also denoted $\sim$, on Ω by $\xi \sim \eta \Leftrightarrow \exists\, \zeta, \zeta' \in \Phi$: $\zeta\eta\zeta'$ is defined and equals ξ. Denote the equivalence classes by $[\xi], \xi \in \Omega$, and the set of them by Ω/Φ. (Note that, $\forall\, x,y \in B$, $x \sim y \Leftrightarrow \tilde{x} \sim \tilde{y}$.)

Then the following defines the structure of a groupoid Ω/Φ with base B/Φ: the source and target projections are $\bar{\alpha}([\xi]) = [\alpha(\xi)]$, $\bar{\beta}([\xi]) = [\beta(\xi)]$, the object inclusion map is $[x] \mapsto \widetilde{[x]} = [\tilde{x}]$, and the product $[\eta][\xi]$, where $\alpha(\eta) \sim \beta(\xi)$, is defined as $[\eta\zeta^{-1}\xi]$, where ζ is any element of Φ with $\alpha\zeta = \alpha\eta$ and $\beta\zeta = \beta\xi$. The inverse of $[\xi]$ is $[\xi^{-1}]$.

The projections $\natural: \xi \mapsto [\xi]$, $\Omega \to \Omega/\Phi$, $\natural_o: x \mapsto [x]$, $B \to B/\Phi$ constitute a groupoid morphism. The kernel of $\natural$ is Φ.

Proof. Exercise for the reader. //

Ω/Φ is the quotient groupoid of Ω over the normal subgroupoid Φ. The notation '$\natural$' should be read as 'natural', for 'natural projection'. Note the extreme cases: $\Omega/\tilde{B}$ is isomorphic to Ω under $\natural$, Ω/Ω is a base groupoid (not necessarily singleton), and $\Omega/G\Omega$ is isomorphic to $\tilde{B}$.

It is easy to see that an injective morphism has the base subgroupoid of its domain as kernel, and that a morphism whose kernel is the base subgroupoid is piecewise-injective. The following example is a surjective morphism whose kernel is the base subgroupoid but which is not (always) an isomorphism.

Example 2.8. Let $P(B,G,\pi)$ be a principal bundle and consider the associated groupoid $\frac{P \times P}{G}$ constructed in 1.10. It is easy to see that the map $P \times P \to \frac{P \times P}{G}$, $(u_2,u_1) \mapsto \langle u_2,u_1 \rangle$ is a morphism of groupoids over $\pi: P \to B$, where $P \times P$ has the trivial groupoid structure of 1.4. The kernel is the diagonal Δ_P of P, which is the base subgroupoid of $P \times P$. //

A surjective group morphism can be factored into the projection onto a quotient group followed by an isomorphism. This example shows that the straightforward generalization to groupoids is not valid. Surjective groupoid morphisms are not determined by their kernels: both the morphism in 2.8 and $id_{P \times P}$ are surjective morphisms with kernel Δ_P. Notice that the contrasting notations Ω/Φ and $\frac{P \times P}{G}$ are used to emphasize that there is no subgroupoid 'G' of $P \times P$ that makes $\frac{P \times P}{G}$ a quotient of groupoids.

For morphisms which are not only surjective, but are also piecewise-surjective, such a straightforward factorization is possible:

Proposition 2.9. Let $\phi: \Omega \to \Omega'$ be a groupoid morphism over $\phi_o: B \to B'$ with kernel Φ.

(i) If ϕ is base-surjective and piecewise-surjective then $\bar{\phi}: \Omega/\Phi \to \Omega'$, $[\xi] \mapsto \phi(\xi)$ is an isomorphism of groupoids and $\phi = \bar{\phi}\circ\natural$.

(ii) If there is an isomorphism of groupoids $\psi: \Omega/\Phi \to \Omega'$ such that $\phi = \psi\circ\natural$, then ϕ is base-surjective and piecewise-surjective.

Proof. (i) Clearly $\bar{\phi}$ is surjective, since ϕ is. Suppose $\bar{\phi}([\xi]) = \bar{\phi}([\eta])$, that is, $\phi(\xi) = \phi(\eta)$. Then $\beta\xi$ and $\beta\eta$ have the same image, say z, under ϕ_o so, since $\phi^{\beta\eta}_{\beta\xi}: \Omega^{\beta\eta}_{\beta\xi} \to \Omega'^{z}_{z}$ is surjective, there is an element $\zeta \in \Omega^{\beta\eta}_{\beta\xi}$ such that $\phi(\zeta) = \tilde{z}$; such an element must actually be in $\Phi^{\beta\eta}_{\beta\xi}$. Similarly there is an element $\zeta' \in \Phi^{\alpha\eta}_{\alpha\xi}$. Now $\zeta^{-1}\eta\zeta'\xi^{-1}$ is defined, is an element of $\Omega^{\beta\xi}_{\beta\xi}$, and is mapped by ϕ to $\tilde{z}$, so it is actually an element of $\Phi^{\beta\xi}_{\beta\xi}$; denote it by λ. Then $\xi = (\zeta\lambda)^{-1}\eta\zeta'$, which shows that $\xi \sim \eta$; that is, $[\xi] = [\eta]$.

(ii) $\natural$ is base-surjective by construction. To prove that $\natural^{y}_{x}: \Omega^{y}_{x} \to (\Omega/\Phi)^{[y]}_{[x]}$ is surjective, take $[\xi] \in \Omega/\Phi$ with $\bar{\beta}([\xi]) = [y]$, $\bar{\alpha}([\xi]) = [x]$. Then $\beta\xi \sim y$, $\alpha\xi \sim x$ so $\exists\ \zeta,\zeta' \in \Phi$ such that $\zeta: y \to \beta\xi$ and $\zeta': x \to \alpha\xi$. Now $\zeta^{-1}\xi\zeta' \sim \xi$ and $\zeta^{-1}\xi\zeta' \in \Omega^{y}_{x}$. //

In the rest of these notes we will be mostly concerned with morphisms $\phi: \Omega \to \Omega'$ for which $B = B'$ and $\phi_o = id_B$, or at any rate for which ϕ_o is a bijection. For morphisms ϕ with ϕ_o a bijection, surjectivity is equivalent to piecewise-surjectivity so 2.9 shows that in this case we have a factorization of an arbitrary morphism into a natural projection followed by an isomorphism followed by an inclusion, exactly as for group morphisms. Two other simplifications are possible in this case: (i) the kernel of a base-bijective morphism (indeed of a base-injective one) is the union of its vertex groups, that is, in the terminology of 3.1, it is a totally intransitive groupoid, and so there is no need to consider the equivalence relation on the base when quotienting over such a kernel, (ii) when quotienting a groupoid Ω over a totally intransitive normal subgroupoid Φ, the relation "$\xi \sim \eta \iff \exists\ \zeta,\zeta' \in \Phi$: $\zeta\eta\zeta'$ is defined and equals ξ" may be defined by "$\xi \sim \eta \iff \exists\ \lambda \in \Phi$: $\eta\lambda$ is defined and equals ξ" (this is a simple consequence of the facts that ζ, η, ζ' must now all belong to the same Ω^{x}_{x}, and Φ^{x}_{x} is a normal subgroup of Ω^{x}_{x}). It is also true for base-bijective morphisms that a morphism is