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PREFACE

This collection of review articles is aimed at those with some background knowledge of Dynamical Systems Theory. It will be useful to graduate students and researchers wishing to familiarize themselves with current research, as well as to those currently working in the field.

Each of the authors has given a survey of an active research topic. The aim is to provide a useful review of the directions in which particular lines of research are going, together with a wide list of references for further reading, and to provide the reader with a number of open problems.

This book is loosely associated with the conference on "Theoretical and Numerical Problems in the Study of Chaotic Ordinary Differential Equations" held at King's College, Cambridge in June and July 1986. The conference was funded by the S.E.R.C. and the Dynamical Systems Project of King's College Research Centre.

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T.J.B.
J.W.S.

INTRODUCTION

In recent years Dynamical Systems has attracted attention from workers in diverse fields. The use of powerful computers and computer graphics in numerical simulations has led to growing interest in "chaos". A wide range of scientists including theoretical physicists, engineers, biologists and ecologists have raised interesting problems which provide new sources of "applied" motivation beyond the traditional questions from classical mechanics. Their interaction with mathematicians has stimulated new lines of research and has been particularly important in determining the new directions taken by Dynamical Systems in the last decade.

The approaches to these new problems have several themes in common. Complicated structures are modelled by deterministic systems with a few variables. The bifurcation patterns of parametrized families of systems are studied. Flows are reduced to Poincaré maps and all systems are modelled by one-dimensional maps whenever possible. In experimental systems, attractors are reconstructed from time series.

Typical questions of interest are to prove the existence of numerically observed "strange attractors" such as that in the Hénon map and to describe the structure of such strange attractors. We would like to understand how these complicated sets can be created from dynamically simple ones through a series of bifurcations. Different kinds of scaling behaviour in strange sets can be found and must be explained. In low dimensional systems the possible range of dynamical behaviour is restricted and so, in principle, should be capable of classification.

The articles in this book are primarily research texts and do not provide a systematic introduction to Dynamical Systems theory. Some books giving a more elementary background are Palis and de Melo [1982], Guckenheimer and Holmes [1983], Collet and Eckmann [1980] and Devaney [1986].

Much of the pure mathematical work in Dynamical Systems over the last twenty years has concentrated on Axiom A systems (first defined by Smale [1967] in his famous survey article) and they are now well understood. (Smale's horseshoe map is the best known example of an Axiom A system.) The use of symbolic dynamics has given a nice description of the nonwandering set in terms of subshifts of finite type and has also led to a good understanding of the ergodic theory of these maps. Smale conjectured that Axiom A diffeomorphisms are open and dense in the space of all diffeomorphisms (so that a typical system would satisfy Axiom A) but this was soon found to be false. For example, neither the Hénon map nor the Lorenz equations are structurally stable (and do not therefore satisfy Axiom A) in the parameter regions where strange attractors have been observed numerically. One of the main problems in Dynamical Systems is to find ways of describing such structurally unstable chaotic systems. The Smale programme has bequeathed a variety of techniques for approaching this problem, for example the usual method of showing that a particular system is chaotic is to prove that it contains a horseshoe. Many systems contain important hyperbolic sets (generalized horseshoes). Glendinning's article in this book describes hyperbolic subsets near to homoclinic orbits. In van Strien's paper on one dimensional maps he describes a result of Mañé which says, roughly, that on compact invariant sets away from the critical point a map of the interval is hyperbolic. Applications of the ergodic theory of Axiom A systems arise in the study of scaling spectra in the renormalisation strange sets discussed by Rand.

In order to obtain detailed descriptions of the behaviour of non-Axiom A systems it is necessary to impose restrictive hypotheses. The most natural kind of assumption to make is to specify the phase space in which the dynamics are taking place because the topology of the phase space is such a severe constraint on possible behaviour. This is why the dynamics of maps of the circle and of the interval are relatively well understood. The development of kneading theory and the use of the Schwarzian derivative, together with the observation of universal behaviour, gave the study of unimodal maps of the interval a great boost in the late 1970's. Van Strien's article in this book gives an account of some of the most sophisticated methods available in the study of one dimensional maps. The constraint of dimension is used to good effect elsewhere. For example, Holmes uses the fact that periodic orbits in three dimensional flows are knots whose knot type is invariant as a parameter is changed. By contrast there are apparently simple questions in low dimensions that remain open. Lloyd's article on the number of limit cycles in a polynomial vector field of the plane is a case in point. Such systems are simple enough that one can prove quite a lot about them, but the original problem (Hilbert's 16th) has remained intractable.

Bifurcation Theory is one of the most powerful techniques in the study of dynamical systems and is used in each of the articles in this collection. Here one attempts to understand a family of systems by concentrating on those parameter values which are at the boundary between different classes of structurally stable systems. The analysis has usually been local in a neighbourhood around a fixed or periodic point, although it has often been observed (experimentally or numerically) that the local analysis holds far from the bifurcation. If two or more parameters are varied then highly degenerate systems can be found which are "organising centres" that enable one to describe a wide range of behaviour with a local analysis. Lloyd's article on limit cycles in polynomial systems uses bifurcation techniques to create new limit cycles by bifurcating from the fixed points at zero and infinity, and from homoclinic loops. The article by Stewart considers the physically important problem of bifurcations in systems with symmetries; symmetry leads to more complicated behaviour, yet at the same time the presence of symmetry can simplify the analysis. Rand's article reviews recent work in which one combines the local approach together with rescaling or renormalisation. The renormalisation group was imported from Physics by Feigenbaum for his analysis of the cascade of period doublings. Renormalisation methods can describe sequences of bifurcations (as opposed to isolated bifurcations) that all occur as a result of some simple mechanism. Sequences of bifurcations also occur (for slightly different reasons) near to homoclinic orbits in 3 or higher dimensional flows. Work on such bifurcations was pioneered by by Shil'nikov and is described in Glendinning's article. A related use of Bifurcation Theory is the attempt to understand complicated dynamics in a strange set by understanding the sequence of bifurcations that occur as it is created in a one-parameter family of systems, starting with something well understood. Holmes aims at this by describing some of the knot types that arise as periodic orbits in strange attractors.

An important part of Hamiltonian Systems is the theory of area preserving twist maps of the annulus, which model the dynamics in a neighbourhood of an elliptic fixed point. The celebrated K.A.M. theorem implies that in such a neighbourhood a set of points with positive Lebesgue measure lie on invariant circles. A major problem here is to understand how these invariant circles break up, and Rand's article discusses an approach to this problem using a renormalisation group analysis. Many long-studied Hamiltonian systems, such as the Kepler problem, have symmetries and these are the subject of the penultimate section of Stewart's article. An introduction to the theory of Hamiltonian Systems can be found in the new book compiled by MacKay and Meiss [1987].

Numerical "experiments" are increasingly important in the study of Dynamical Systems. The pioneering studies of Lorenz [1963], Hénon and Heiles [1964], and Hénon [1978] have inspired a huge amount of numerical work which has aided our intuition and motivated numerous theoretical studies. Numerical investigations to discover new theorems are now standard, and the results have been dramatic. Computer experiments with the logistic map pointed the way to many results which are described in van Strien's and Rand's articles. It is modern folklore that Feigenbaum's discovery of the universal scaling in period doubling was made using a hand calculator. Glendinning's paper here discusses several problems arising from numerical observations of bifurcations in the Lorenz equations. Stewart's article is motivated in part by physical systems such as Taylor-Couette flow, and numerical simulations of partial differential equations which model these symmetric systems. A different use of computers - untiring and accurate algebraic manipulation - is used extensively in Lloyd's study of planar polynomial systems.

We hope that the articles in this book will give a flavour of some of the new directions in Dynamical Systems. A whole range of techniques have been adopted to provide descriptions of chaotic and non-chaotic systems. For chaotic systems these include scaling properties of fractal invariant sets, descriptions of persistent structures such as hyperbolic subsets and knotted periodic orbits, and finding sequences of bifurcations that create chaotic behaviour. None of these techniques provide a single satisfactory description of a chaotic system, for the moment a piecemeal approach is all that is possible. For non-chaotic systems the open problems are just as difficult. The techniques of Bifurcation Theory are of some help, but do not really provide a view of the global behaviour of these systems. The wealth of behaviour in even the most simple systems is enough to keep us all intrigued for many years to come.

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References

Collet, P. and Eckmann, J.-P. (1980). *Iterated maps of the Interval as Dynamical Systems*, Progress in Physics, Vol 1, Birkhäuser-Boston.

Devaney, R.L. (1986). *An Introduction to Chaotic Dynamics*, Benjamin/Cummings, Menlo Park, CA.

Guckenheimer, J. and Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Appl. Math. Sci. Series, Vol. 42, Springer-Verlag, New York.

Hénon, M. (1976). A Two-Dimensional Mapping with a Strange Attractor. *Comm. Math. Phys.*, **50**, 69-77.

Hénon, M. and Heiles, C. (1964). The Applicability of the Third Integral of Motion: Some Numerical Experiments. *Astron. J.*, **69**, 73-79.

Lorenz, E.N. (1963). Deterministic Non-periodic Flows. *J. Atmos. Sci.*, **20**, 130-141.

MacKay, R.S. and Meiss, J.D. (1987). *Hamiltonian Dynamical Systems*. Adam Hilger, Bristol.

Palis, J. and de Melo, W. (1982). *Geometric theory of Dynamical Systems: An Introduction*. Springer-Verlag, New York.

Smale, S. (1967). Differentiable Dynamical Systems. *Bull. Amer. Math. Soc.* **73**, 747-817.

