Introduction

In this book we present a comprehensive introduction to the use of Clifford algebras and Dirac operators in harmonic analysis and analysis more generally. In the past 30 years, Clifford algebras and Dirac operators have played a key role in three of the most important areas of mathematical research during that time: the boundedness of the Cauchy integral on Lipschitz surfaces, the realization of discrete series representations of semi-simple Lie groups, and the celebrated Atiyah–Singer index theorem. Much as an analyst would like to understand and appreciate these developments, however, there are formidable technical barriers to doing so, particularly for more classically trained analysts, as we have found to our cost over the years. Thus our aim from the outset has been to meld into a coherent and reasonably self-contained whole a body of ideas from classical singular integral theory, representation theory and analysis on manifolds, with a view to making this material accessible to more classically trained analysts.

Now the starting point for much of classical harmonic analysis is the study of the boundary regularity of harmonic functions in domains in Euclidean space. Classical Hardy space theory explores the consequences of the improved boundary regularity obtained when consideration is restricted to *analytic* functions in the plane. On the other hand, for $SL(2, \mathbb{R})$, the starting-point for representation theory of semi-simple Lie groups, some important unitary representations become irreducible only on restriction to *analytic* functions. It may be a dramatic overstatement to characterize analytic functions as those in the kernel of a first-order elliptic differential operator – the Cauchy–Riemann $\overline{\partial}$ operator – which factors the Laplacian and has rotation-invariant symbol; but it is precisely such properties that one looks for in differential operators on more general manifolds. For one can then develop a Hardy H^p theory on Euclidean space including an analysis of elliptic boundary value problems, as well as explicit realizations of semi-simple Lie groups on associated symmetric spaces. Index theorems arise in both cases, of course. Dirac operators and their generalization, the so-called *operators of Dirac type*, have such properties.

Much earlier, quite independently of all these analytic ideas, Clifford introduced his algebras as a common generalization of Grassmann's exterior algebra and Hamilton's quaternions, both of which sought to capture the geometric and algebraic properties of Euclidean space. Indeed, Clifford used the name 'geometric algebras' for his algebras quite appropriately, because the universal Clifford algebra for $\mathbb{R}^n$ is the minimal enlargement of $\mathbb{R}^n$ to an associative algebra capturing precisely the algebraic, geometric and metric properties of Euclidean space. It is not surprising, therefore, that the bundle formed by the Clifford algebra of the tangent space at each point of a manifold should be so important in the geometric analysis of that manifold.

In chapter 1 we present the general theory of Clifford algebras in an elementary and thoroughgoing fashion, which should be accessible to the algebra 'neophyte'; it is our aim to give a coherent account of material which is presently scattered throughout the literature with no one account being readily accessible. In chapter 2 we quickly review the classical Hardy space theory and its extension to minimally smooth domains, and then develop a higher-dimensional analogue for this theory based upon functions in the kernel of the Dirac operator. In chapter 3 we explore further the connections between Clifford algebras and representations of the spin group and of the rotation group. Then, in chapter 4, we define a more general notion of *operators of Dirac type*, and show that all of the important rotation-invariant geometric differential operators of Euclidean analysis are in fact of Dirac type. Finally, in chapter 5 we introduce and then study Clifford algebras and Dirac operators on more general manifolds, concluding with a recent simplified proof of the local Atiyah–Singer index theorem.

This book had its beginnings in the fall of 1985, when one of us (M.M.) was a visiting faculty member at the University of Texas at Austin. To whatever extent we have succeeded in our goal, we owe a debt of thanks to many of our friends and colleagues who have made this success possible. In particular, we wish to thank René Beerends, Klaus Bichteler, Chris Meaney and John Ryan for numerous helpful discussions, but most of all we wish to thank Kathy Davis, Gene Fabes and Ray Kunze for immeasurable help in the formulation of ideas going into the book, as well as in the writing of the book. One of us (M.M.) would like to acknowledge the particular help and support of her good friends and colleagues Daniel Farkas, and Carol and Frank Burch-Brown, without whom this work might never have come to fruition. Partial support from the National Science Foundation is acknowledged by both of us, too.

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Clifford algebras

Associated with any Euclidean space $\mathbb{R}^n$ or Minkowski space $\mathbb{R}^{p,q}$ is a universal Clifford algebra, denoted by $\mathfrak{A}_n$ and $\mathfrak{A}_{p,q}$, respectively. Roughly speaking, a Clifford algebra is an associative algebra with unit into which a given Euclidean or Minkowski space may be embedded, in which the corresponding quadratic form may be expressed as the negative of a square. The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$ are the simplest examples.

Our intent in this chapter is to give an elementary, coherent, and largely self-contained account of the theory of Clifford algebras. In sections 1 and 2 we present the definitions basic to all of our work. The balance of section 2 is devoted to three constructive proofs of the existence of universal Clifford algebras: two basis-free constructions using tensor algebras and exterior algebras, and a basis-dependent construction. The reader who is willing to accept the existence of Clifford algebras may wish to proceed directly to the statement of the major structural results in section 3. Sections 4, 5, and 6 explore the interconnections between Clifford algebras and orthogonal groups; the spin representation and spin groups will be studied in detail, with $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}(p,q+1)$ both being realized in $\mathfrak{A}_{p,q}$ using the notion of transformers. The reader who is primarily interested in the analytic applications of Clifford algebras may wish to proceed directly to the discussion of the Euclidean case in section 7. Section 8 is a discussion of spin groups as Lie groups. In section 9 we construct various realizations of $\mathrm{Spin}(p,q)$, $p+q \leq 6$, whereby these groups are explicitly identified with classical Lie groups.

1 Quadratic spaces

Let V be a finite-dimensional vector space over the scalar field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *quadratic form* on V is a mapping $Q : V \to \mathbb{F}$ such that

(1.1)(i) $Q(\lambda v) = \lambda^2 Q(v)$, $\lambda \in \mathbb{F}$, $v \in V$,

(1.1)(ii) *the associated form*

$$B(v,w) = \tfrac{1}{2}\{\, Q(v) + Q(w) - Q(v-w) \,\} , \qquad v, w \in V ,$$

is bilinear.

When such a Q exists, the pair (V, Q) is said to be a *quadratic space*; every vector space over $\mathbb{F}$ becomes a quadratic space with respect to the trivial quadratic form $Q \equiv 0$, for instance. The significance of condition (1.1)(ii) is that the form B defines an inner product on $V \times V$, and so all the usual geometric properties of inner product spaces can be exploited. Typically, a quadratic space arises from an inner product space, defining Q on V by, say, $Q(v) = (v \mid v)$ where $(\cdot \mid \cdot)$ is the inner product on $V \times V$. For example, if $(\cdot \mid \cdot)$ is the usual Euclidean inner product on $\mathbb{R}^n$ and $|v|^2 = (v \mid v)$, then both $(\mathbb{R}^n, |\cdot|^2)$ and $(\mathbb{R}^n, -|\cdot|^2)$ are real quadratic spaces with associated bilinear forms $(\cdot \mid \cdot)$ and $-(\cdot \mid \cdot)$ respectively. More generally, let p, q be non-negative integers with $p + q > 0$ and define a pseudo-Euclidean or Minkowski quadratic form on $\mathbb{R}^{p+q}$ by

(1.2)

$$Q_{p,q}(u) = -(u_1^2 + \cdots + u_p^2) + (u_{p+1}^2 + \cdots + u_{p+q}^2) , \quad u = (u_1, \ldots, u_{p+q}) ;$$

the corresponding real quadratic space we shall call *Minkowski* space and denote it by $(\mathbb{R}^{p,q}, Q_{p,q})$. Clearly $(\mathbb{R}^{n,0}, Q_{n,0})$ reduces to $(\mathbb{R}^n, -|\cdot|^2)$, while $(\mathbb{R}^{0,n}, Q_{0,n})$ is just $(\mathbb{R}^n, |\cdot|^2)$. By convention, $\mathbb{R}^{0,0} = \{0\}$. In the complex case, $(\mathbb{C}^n, Q_n)$ becomes a complex quadratic space on setting

(1.3) $$Q_n(z) = z_1^2 + \cdots + z_n^2 , \qquad z = (z_1, \ldots, z_n) ;$$

note that in (1.3) the associated form $B_n(z, w) = z_1 w_1 + \cdots + z_n w_n$ is complex linear in w, not conjugate-linear as in the usual inner product on $\mathbb{C}^n$.

Now let (V, Q) be an arbitrary quadratic space and $\{e_j\}$ a basis for V. Then

$$Q(v) = \sum_{j,k} B(e_j, e_k) v_j v_k , \qquad v = \sum_j v_j e_j ,$$

and if there is a basis which is B-*orthogonal* in the sense that $B(e_j, e_k) = 0$ when $j \neq k$, the expression for $Q(v)$ reduces to diagonal form

$$Q(v) = \sum_j Q(e_j) v_j^2 , \qquad v = \sum_j v_j e_j .$$

Such a basis is easily constructed. Let

$$\mathrm{Rad}(V,Q) = \{\ w \in V : B(v,w) = 0\ ,\ \text{all } v \in V\ \} = V^{\perp}$$

be the *radical* of (V,Q). We say that (V,Q) is *non-degenerate* if $\mathrm{Rad}(V,Q) = \{0\}$; otherwise (V,Q) is *degenerate*, in which case V can be written as the B-orthogonal direct sum

$$V = \mathrm{Rad}(V,Q) \oplus \mathrm{Rad}(V,Q)^{\perp}$$

of $\mathrm{Rad}(V,Q)$ and its B-orthogonal complement. Clearly $(\mathrm{Rad}(V,Q)^{\perp}, Q)$ is non-degenerate, and B-orthogonal bases $\{e_j\}$, $Q(e_j) \neq 0$, can be constructed in the usual way for $\mathrm{Rad}(V,Q)^{\perp}$, or for V if (V,Q) is already non-degenerate. Using (1.1)(i) to normalize the e_j, we can also assume that $Q(e_j) = \pm 1$ when $\mathbb{F} = \mathbb{R}$, while $Q(e_j) = 1$ when $\mathbb{F} = \mathbb{C}$. Now augment this basis by any basis of $\mathrm{Rad}(V,Q)$ if (V,Q) is degenerate. Since Q is trivial on $\mathrm{Rad}(V,Q)$, we thus obtain a basis $\{e_j\}$ for V such that

(1.4)

(i) $B(e_j, e_k) = 0\ ,\quad j \neq k\ ,$

(ii) $\{e_j : Q(e_j) = 0\}$ *is a basis for* $\mathrm{Rad}(V,Q)\ ,$

(iii) $\{e_j : Q(e_j) \neq 0\}$ *is a basis for* $\mathrm{Rad}(V,Q)^{\perp}$ *such that* $Q(e_j) = \pm 1$ *when* $\mathbb{F} = \mathbb{R}$, *while* $Q(e_j) = 1$ *when* $\mathbb{F} = \mathbb{C}$.

With some abuse of customary terminology, a basis for V satisfying (1.4)(i), (ii), (iii) will be said to be a *normalized basis*; many of the algebraic constructions to be discussed are conveniently given using such a basis. For instance, from such a basis it follows that every quadratic space (V,Q) is the sum of the particular examples given already. More precisely, we have the following.

(1.5) Theorem.
Let (V,Q) be a quadratic space with B-orthogonal decomposition

$$V = \mathrm{Rad}(V,Q) \oplus \mathrm{Rad}(V,Q)^{\perp}.$$

Then

(a) $Q \equiv 0$ *on* $\mathrm{Rad}(V,Q)$,

(b) when $\mathbb{F} = \mathbb{R}$, $(\mathrm{Rad}(V,Q)^{\perp}, Q)$ *is isomorphic to* $\mathbb{R}^{p,q}$ *where* p, q *depend only on* Q,

(c) when $\mathbb{F} = \mathbb{C}$, $(\mathrm{Rad}(V,Q)^{\perp}, Q)$ *is isomorphic to* $(\mathbb{C}^n, Q_n)$ *where* n *depends only on* Q.

Proof. Part (a) is an immediate consequence of the definition of

$\mathrm{Rad}(V,Q)$. If $\mathrm{Rad}(V,Q) \neq V$, choose a basis $\{e_j\}$ satisfying (1.4)(i), (ii), (iii). In the case $\mathbb{F} = \mathbb{R}$ this basis can be indexed so that

$$Q(u) = Q\Big(\sum_j u_j e_j\Big) = -(u_1^2 + \cdots + u_p^2) + (u_{p+1}^2 + \cdots + u_{p+q}^2) ,$$

and $\dim(V,Q)^{\perp} = p + q > 0$; by Sylvester's theorem, the values of p, q do not vary with the choice of basis. Part (b) is now clear, and part (c) is proved in the same way. ∎

2 Clifford algebras

As Clifford's paper introducing 'geometric algebras' shows, Clifford based his ideas on the common features he saw in the construction of Grassmann's algebra and Hamilton's quaternions. In the framework of modern algebra we shall derive both constructions simultaneously beginning with an arbitrary quadratic space (V, Q), V a finite-dimensional vector space over $\mathbb{F}$. Let $\mathbf{A}$ be an associative algebra over $\mathbb{F}$ with identity 1 and $\nu : V \to \mathbf{A}$ an $\mathbb{F}$-linear embedding of V into $\mathbf{A}$.

(2.1) Definition.

The pair $(\mathbf{A}, \nu)$ is said to be a Clifford algebra for (V, Q) when

(i) *$\mathbf{A}$ is generated as an algebra by $\{\nu(v) : v \in V\}$ and $\{\lambda 1 : \lambda \in \mathbb{F}\}$,*

(ii) *$(\nu(v))^2 = -Q(v)1$, all $v \in V$.*

Roughly speaking, therefore, condition (ii) ensures that $\mathbf{A}$ is an algebra in which there exists a 'square root' of the quadratic form $-Q$; condition (i) is a minimality restriction on the 'size' of $\mathbf{A}$.

Some simple examples illustrate how this definition contains the algebras whose structure prompted Clifford to introduce 'geometric algebras'.

(2.2) Examples.

(i) When $Q \equiv 0$ on V, let $\mathbf{A}$ be the exterior algebra $\Lambda^*(V) = \sum_{k=0}^{n} \Lambda^k(V)$ with $n = \dim V$, $\Lambda^0(V) \cong \mathbb{F}$, and $\Lambda^1(V) \cong V$, and let $\nu : V \to \Lambda^1(V)$. Since every element of $\Lambda^k(V)$, $k \geq 2$, is of the form $v_1 \wedge \cdots \wedge v_k$, clearly $\Lambda^*(V)$ is generated by $\{\nu(v) : v \in V\}$ and $\{\lambda 1 : \lambda \in \mathbb{F}\}$; in addition,

$$(\nu(v))^2 = v \wedge v = 0 = -Q(v)1 .$$

Hence the Grassmann algebra $\Lambda^*(V)$ is a Clifford algebra for (V, Q) when $Q \equiv 0$.

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(ii) Define the *Pauli matrices* in $\mathbb{C}^{2\times 2}$ by

(2.3)
$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and *associated Pauli matrices* by

(2.4)
$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

As elements of the associative algebra $\mathbb{C}^{2\times 2}$,

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I, \qquad e_0^2 = I, \qquad e_1^2 = e_2^2 = e_3^2 = -I,$$

while

(2.5)
$$\sigma_j\sigma_k = -i\sigma_\ell, \qquad e_je_k = e_\ell$$

when $\{j,k,\ell\}$ is a cyclic permutation of $\{1,2,3\}$. These matrices will occur throughout the theory of Clifford algebras. For instance, set

$$\mathfrak{A}_{0,0} = \{\lambda\sigma_0 : \lambda \in \mathbb{R}\},$$

$$\mathfrak{A}_{1,0} = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}, \qquad \mathfrak{A}_{0,1} = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbb{R} \right\},$$

and

$$\mathfrak{A}_{0,2} = \left\{ \begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{bmatrix} : x_j \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} : z_j \in \mathbb{C} \right\}.$$

Then each of these is an associative subalgebra (over $\mathbb{R}$) of $\mathbb{C}^{2\times 2}$ having an identity, and

(2.6)
$$\mathfrak{A}_{0,0} \cong \mathbb{R}, \quad \mathfrak{A}_{1,0} \cong \mathbb{R} \oplus \mathbb{R}, \quad \mathfrak{A}_{0,1} \cong \mathbb{C}, \quad \mathfrak{A}_{0,2} \cong \mathbb{H}$$

where $\mathbb{H}$ is Hamilton's algebra of quaternions. As the notation suggests, $\mathfrak{A}_{p,q}$ also is a Clifford algebra for $\mathbb{R}^{p,q}$ with respective embeddings ν given by

$$0 \to 0, \quad y \to y\sigma_3, \quad y \to ye_2, \quad (x_1, x_2) \to x_1e_1 + x_2e_2.$$

In each case the proof amounts to a simple computation using properties of the σ_j and e_j. For instance, in $\mathfrak{A}_{0,2}$

$$\big(\nu(x_1, x_2)\big)^2 = (x_1e_1 + x_2e_2)^2 = -(x_1^2 + x_2^2)e_0$$

since the cyclic permutation property of $\{e_1, e_2, e_3\}$ ensures that $e_1e_2 + e_2e_1 = 0$; alternatively, by direct calculation we see that

$$\begin{bmatrix} ix_1 & x_2 \\ -x_2 & -ix_1 \end{bmatrix}^2 = -(x_1^2 + x_2^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(iii) As an associative algebra over $\mathbb{C}$, the matrix algebra $\mathbb{C}^{2\times 2}$ is a

Clifford algebra for $(\mathbb{C}^2, Q_2)$ with

$$\nu : \mathbb{C}^2 \longrightarrow \mathbb{C}^{2\times 2}\,, \qquad \nu : (z_1, z_2) \longrightarrow \begin{bmatrix} 0 & z_1 - iz_2 \\ z_1 + iz_2 & 0 \end{bmatrix}\,.$$

Again the proof is just a simple computation.

Some elementary properties of a Clifford algebra follow readily from its definition. Let (V, Q) be a quadratic space of dimension n and $(\mathbf{A}, \nu)$ a Clifford algebra for (V, Q). It is very convenient not to make distinction between $\lambda \in \mathbb{F}$ and $\lambda 1 \in \mathbf{A}$, or between $v \in V$ and $\nu(v) \in \mathbf{A}$. With this understanding, $\mathbf{A}$ is generated by $\mathbb{F}$ and V; in addition $v^2 = -Q(v)$. Similarly, if W is a subspace of V, then the subalgebra of $\mathbf{A}$ generated by $\mathbb{F}$ and W is a Clifford algebra for (W, Q). On the other hand, for arbitrary u, v in V

$$\begin{aligned}(u+v)^2 = -Q(u+v) &= -2B(u,v) - Q(u) - Q(v)\\ &= -2B(u,v) + u^2 + v^2\ ;\end{aligned}$$

but, by direct expansion,

$$(u+v)^2 = u^2 + uv + vu + v^2\ .$$

Thus the algebra structure on $\mathbf{A}$ enables us to express the inner product B on $V \times V$ as

$$B(u,v) = -\tfrac{1}{2}(uv + vu)\ ;$$

in particular,

(2.7) $$e_j e_k + e_k e_j = -2Q(e_j)\delta_{jk}\,, \qquad 1 \le j, k \le n\ .$$

for any normalized basis $\{e_j\}_{j=1}^n$ of V. Often a Clifford algebra is defined as being the algebra generated by elements $e_1, \ldots, e_n$ satisfying (2.7).

(2.8) Theorem.

Let $(\mathbf{A}, \nu)$ be a Clifford algebra for (V, Q) and $\{e_j\}_{j=1}^n$ a normalized orthogonal basis of V. Then

(i) $\mathbf{A}$ is spanned by all products

$$e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n}\,, \qquad m_j = 0, 1,$$

where $e_1^0 \cdots e_n^0$ is interpreted as the identity in $\mathbf{A}$,

(ii) $\mathbf{A}$ has dimension at most 2^n.

Proof of 2.8(i) In view of (2.7)

(2.9) $$e_j e_k + e_k e_j = 0 \quad (j \ne k)\,, \qquad e_j^2 = -Q(e_j)\ .$$

Thus any product of powers of the e_j can be reduced to a scalar multiple of $e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n}$ where $m_j = 0$ or 1 and $e_1^0 e_2^0 \cdots e_n^0$ is interpreted as the identity in **A**. The linear span of all such reduced products must then be **A**, since **A** is generated as an algebra by V and $\mathbb{F}$.

These reduced products are very convenient to use both in developing the general theory of Clifford algebras and in studying particular examples, so it is worth having a simple notation for them. Let $N_V = \{1, 2, \ldots, \dim V\}$ and for each non-empty subset α of N_V set

(2.10)
$$e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k}\ , \qquad \alpha = \{\alpha_1, \ldots, \alpha_k\}\ , \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq \dim V\ ;$$

by convention, $e_\emptyset$ is the identity 1 in **A** when $\emptyset$ is the empty subset of N_V.

Proof of 2.8(ii). Since there are 2^n subsets of N_V, **A** is spanned by the corresponding 2^n reduced products. Thus $\dim \mathbf{A} \leq 2^n$. ∎

A Clifford algebra can have maximum dimension $2^{\dim V}$. For instance, all the examples in (2.2) have this property. Indeed, it is well-known that the exterior algebra $\Lambda^*(V)$ has dimension 2^n when $\dim V = n$, and that the reduced wedge products

(2.11)
$$e_\emptyset = 1\ , \quad e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}\ , \quad \alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq N_V,$$

form a basis for $\Lambda^*(V)$. Each of the $\mathfrak{A}_{p,q}$ in (2.6) also has maximum dimension as a (real) Clifford algebra for $\mathbb{R}^{p,q}$, while $\mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C}^{2\times 2}$ have maximum dimension as (complex) Clifford algebras for $(\mathbb{C}^1, Q_1)$ and $(\mathbb{C}^2, Q_2)$ respectively. On the other hand, $\mathfrak{A}_{0,2}$ is a Clifford algebra for $\mathbb{R}^{0,3}$ defining ν on $\mathbb{R}^{0,3}$ by

$$\nu : \mathbb{R}^{0,3} \to \mathfrak{A}_{0,2}\ , \qquad \nu(x_1, x_2, x_3) = x_1 e_1 + x_2 e_2 + x_3 e_3\ ;$$

as before, the anticommutation property $e_j e_k + e_k e_j = 0$, $j \neq k$, ensures that

$$\big(\nu(x_1, x_2, x_3)\big)^2 = -(x_1^2 + x_2^2 + x_3^2) e_0\ .$$

But for this example $\dim \mathbf{A} = (1/2)2^{\dim V}$. In fact, as we shall see later in section 3, the dimension of any Clifford algebra **A** for $\mathbb{R}^{p,q}$ will satisfy exactly one of

(2.12)(a) $\dim \mathbf{A} = 2^{p+q}$,

(2.12)(b) $\dim \mathbf{A} = (1/2)2^{p+q}$, $p - q - 1 \equiv 0 (\mathrm{mod}\, 4)$, $p + q$ *is odd, and* $e_1 e_2 \cdots e_{p+q} \in \mathbb{R}$ *for some normalized basis* $\{e_1, \ldots, e_{p+q}\}$ *of* $\mathbb{R}^{p,q}$

The Clifford algebra $\mathfrak{A}_{0,2}$ for $\mathbb{R}^{0,3}$ falls into category (2.12)(b).

Clifford algebras of maximum dimension have an important distinguishing property.