ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Factorization calculus and geometric probability

R. V. AMBARTZUMIAN

Institute of Mathematics, Armenian Academy of Sciences

CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York Port Chester Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1990

Printed in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data
Ambartzumian, R. V. (Ruben V.), *1941–*
Factorization calculus and geometric probability.
1. Integral geometry
I. Title II. Series
516.3′62

Library of Congress cataloguing in publication data
Ambartzumian, R. V.
Factorization calculus and geometric probability/R. V. Ambartzumian.
p. cm.–(Encyclopedia of mathematics and its applications; v. 33)
ISBN 0 521 34535 9
1. Stochastic geometry. 2. Geometric probabilities.
3. Factorization (Mathematics) I. Title. II. Series.
QA273.5.A45 1990
519.2–dc20 89-7314 CIP

ISBN 0 521 34535 9 hardback

AS

CONTENTS

1

Cavalieri principle and other prerequisites

The aim of this chapter is to present some basic mathematical tools on which many constructions in the subsequent chapters depend.

Thus we will often refer to what we call the 'Cavalieri principle'. We try to revive this old familiar name because of the surprising frequency with which the transformations Cavalieri considered about 350 years ago occur in integral geometry.

No less useful will be the principles which we call 'Lebesgue factorization' and 'Haar factorization'. The first is a rather simple corollary of a well-known fact that in $\mathbb{R}^n$ there is only one (up to a constant factor) locally-finite measure which is invariant with respect to shifts of $\mathbb{R}^n$, namely the Lebesgue measure. Haar factorization is a similar corollary of a much more general theorem of uniqueness of Haar measures on topological groups. We use the two devices in the construction of Haar measures on groups starting from Haar measures on subgroups.

Integral geometry binds together such notions as metrics, convexity and measures, and these interconnections remain significant throughout the book; §§1.7 and 1.8 are introductory to this topic.

1.1 The Cavalieri principle

The classical Cavalieri principle in two dimensions can be formulated as follows.

Let D_1 and D_2 be two domains in a plane (see fig. 1.1.1).

If for each value of y the length of the chords X_1 and X_2 coincide, then the areas of D_1 and D_2 are equal.

The proof of this beautiful geometrical proposition follows from the representation of the area of D_i, $i = 1, 2$, by the integral

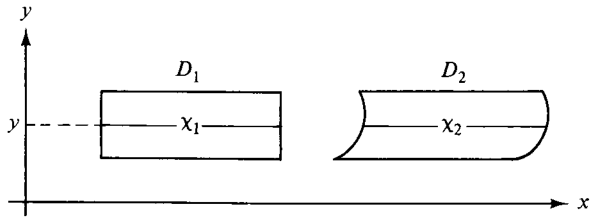

Figure 1.1.1 X_i is the intersection of D_i with the horizontal line on the level y

$$\int X_i \, dy.$$

Pairs of domains having the above property arise whenever we consider transformations of the plane of the type

$$x_1 = x + a(y)$$
$$y_1 = y,$$

which are clearly area-preserving. To these transformations the following interpretation can be given.

We consider the plane as composed of 'rigid' horizontal lines. The transformation

$$(x, y) \to (x_1, y_1)$$

rigidly shifts each horizontal line along itself (although the shifts can be different for different lines). The domain D_2 in the above example can be considered to be the image of D_1 under some transformation of this type. We call this a *Cavalieri transformation.* Measures in $\mathbb{R}^2$ which remain invariant with respect to Cavalieri transformations are numerous. For instance, measures given by the densities of the form

$$f(y) \, dx \, dy$$

are all invariant as shown by the identity

$$\iint_{D_2} f(y) \, dx \, dy = \int f(y) X_1(y) \, dy = \iint_{D_1} f(y) \, dx \, dy, \tag{1.1.1}$$

where D_1 and D_2 are as in fig. 1.1.1. Similarly (Fubini theorem) it can be shown that *any* product measure

$$m \times L_1,$$

where L_1 is the Lebesgue measure on the Ox axis and m is *any* measure on the Oy axis, is invariant with respect to Cavalieri transformations of $\mathbb{R}^2$.

In the sequel similar transformations of other product spaces will occur. The typical situation here will be as follows.

Let the basic space $\mathbb{X}$ be a product of two spaces

$$\mathbb{X} = \mathbb{Y} \times \mathbb{R}^k, \quad (x \in \mathbb{X}, y \in \mathbb{Y}, \mathscr{P} \in \mathbb{R}^k)$$

(the second factor is k-dimensional Euclidean).

Assume that a transformation of the space $\mathbb{X}$ has two properties:

(a) it sends each generator, i.e. the set

$$y \times \mathbb{R}^k = \{(y, \mathscr{P}) : y \text{ is constant}, \mathscr{P} \text{ changes in } \mathbb{R}^k\},$$

into the same generator;

(b) it preserves the distances between the points of a generator (i.e. the generators are 'rigid').

Such transformations we again call Cavalieri. The *Cavalieri principle* in this situation is as follows.

Every product measure $m \times L_k$, where L_k is the Lebesgue measure on $\mathbb{R}^k$ and m is any measure on $\mathbb{Y}$, is invariant with respect to Cavalieri transformations of $\mathbb{X}$.

In the spaces of importance described in chapter 2 we actually have natural groups of Cavalieri transformations.

1.2 Lebesgue factorization

We consider a product of two spaces

$$\mathbb{X} = \mathbb{Y} \times \mathbb{R}^k,$$

where $\mathbb{R}^k$ is k-dimensional Euclidean space, while the space $\mathbb{Y}$ here remains unspecified (any separable metric space will suffice).

Let $\mathbb{T}_k$ be the group of translations of $\mathbb{R}^k$. We *define* the action of a translation $t \in \mathbb{T}_k$ on the space $\mathbb{X}$ as follows.

For $(y, \mathscr{P})$, where $y \in \mathbb{Y}, \mathscr{P} \in \mathbb{R}^k$, we put

$$t(y, \mathscr{P}) = (y, t\mathscr{P})$$

(this means that t is y-preserving).

A measure μ on $\mathbb{X}$ is called *invariant with respect to* $\mathbb{T}_k$ (or simply $\mathbb{T}_k$-invariant) if for every $t \in \mathbb{T}_k$ and $C \in \mathbb{X}$ we have

$$\mu(tC) = \mu(C), \quad \text{where} \quad tC = \{t(y, \mathscr{P}) : (y, \mathscr{P}) \in C\}. \tag{1.2.1}$$

To check (1.2.1) it is enough to consider the product sets, i.e. to take

$$C = A \times B, \quad A \subset \mathbb{Y}, \quad B \subset \mathbb{R}^k,$$

in which case (1.2.1) reduces to

$$\mu(t(A \times B)) = \mu(A \times tB) = \mu(A \times B),$$

where tB denotes the translation of B by t:

$$tB = \{t\mathscr{P} : \mathscr{P} \in B\}.$$

As usual, measures which are finite on compact sets we call locally-finite. We say that a measure m *on* $\mathbb{X}$ has a locally-finite projection on $\mathbb{R}^k$, if for every compact $B \subset \mathbb{R}^k$ we have

$$m(\mathbb{Y} \times \mathbb{B}) < \infty.$$

The Lebesgue factorization principle states that:

Any locally-finite and $\mathbb{T}_k$-invariant measure μ on $\mathbb{X} = \mathbb{Y} \times \mathbb{R}^k$ is necessarily a product measure:

$$\mu = m \times L_k,$$

where L_k is Lebesgue measure on $\mathbb{R}^k$ and m is a locally-finite measure on $\mathbb{Y}$. If additionally m has a locally-finite projection on $\mathbb{R}^k$, then

$$\mu = \lambda \cdot P \times L_k,$$

where $\lambda \geqslant 0$ is a constant and P is a probability measure on $\mathbb{Y}$, *i.e.* $P(\mathbb{Y}) = 1$.

Proof Let us fix a set $A_0 \subset \mathbb{Y}$ which has compact closure and let us regard $\mu(A_0 \times B)$ as a set function depending on B. It follows from the properties of μ that this is a measure on $\mathbb{Y}$ which is translation-invariant and locally-finite. It is known from analysis that any such measure is proportional to Lebesgue measure, i.e.

$$\mu(A_0 \times B) = m(A_0) \cdot L_k(B).$$

So far $m(A_0)$ has been some constant which does not depend on B, but may depend on our choice of A_0.

Now we fix a set $B_0 \subset \mathbb{R}^k$ which has compact closure and consider

$$\mu(A \times B_0) = m(A) \cdot L_k(B_0)$$

as a function of A. Clearly $\mu(A \times B_0)$ is a measure on $\mathbb{Y}$. This implies that m is a locally-finite measure on $\mathbb{Y}$. This proves the first assertion. In the case where μ has a locally-finite projection on $\mathbb{R}^k$, we have

$$\mu(\mathbb{Y} \times B_0) = m(\mathbb{Y}) \cdot L_k(B_0) < \infty;$$

i.e.

$$m(\mathbb{Y}) < \infty.$$

We get the second assertion when we put (assuming $\lambda > 0$)

$$\lambda = m(\mathbb{Y}), \quad P = \lambda^{-1} m.$$

In the factorization table 2.8.1 we give several important examples where Lebesgue factorization is directly applied.

We mention, however, that the factorizations of table 2.9.1 are valid under quite different conditions: in the corresponding spaces the group of shifts no longer transforms product sets into product sets.

Remark on terminology In this book we consider only locally-finite measures. However, in the text we often omit the adjective 'locally-finite'. Thus a 'measure' will always mean a 'locally-finite measure'.

1.3 Haar factorization

The main idea of Lebesgue factorization can be extended to product spaces where one of the space factors is a group.

For a broad class of locally-compact topological groups (an exact account of the theory can be found in [4]) an important theorem is valid which establishes existence and uniqueness (up to a constant factor) of the so-called *left-invariant* and *right-invariant Haar measures*. In general the two measures need not be proportional. When they are, we have the *bi-invariant* Haar measure (which is again defined up to a constant factor).

We will always tacitly assume that our groups belong to the class mentioned above, as do all the concrete groups we consider in this book.

Let $\mathbb{U}$ be a group. A non-zero measure h on $\mathbb{U}$ is called *left-invariant Haar* if

$$h(uA) = h(A) \tag{1.3.1}$$

for arbitrary $u \in \mathbb{U}$ and $A \subset \mathbb{U}$. Here

$$uA = \{uu_1 : u_1 \in A\},$$

where uu_1 denotes group multiplication.

A non-zero measure h is called *right-invariant Haar* if

$$h(Au) = h(A) \tag{1.3.2}$$

for any $A \subset \mathbb{U}$ and $u \in \mathbb{U}$. Here

$$Au = \{u_1 u : u_1 \in A\}.$$

We will use the notation $h_{\mathbb{U}}^{(l)}$, $h_{\mathbb{U}}^{(r)}$ and $h_{\mathbb{U}}$, respectively, for left-, right- and bi-invariant measures on $\mathbb{U}$.

By essentially repeating the proof of the previous section we can extend its result to product spaces

$$\mathbb{X} = \mathbb{Y} \times \mathbb{U},$$

where the factor $\mathbb{U}$ is a group (it replaces $\mathbb{R}^k$), $\mathbb{Y}$ again is a separable metric space.

Any measure μ on $\mathbb{X}$ which is invariant with respect to the transformations

$$u_1(y, u) = (y, u_1 u)$$

necessarily factorizes:

$$\mu = m \times h_{\mathbb{U}}^{(l)}, \tag{1.3.3}$$

where m is some measure on $\mathbb{Y}$.

If we use right-multiplication, i.e.

$$u_1(y, u) = (y, uu_1),$$

then the right-invariant Haar measure $h_{\mathbb{U}}^{(r)}$ will appear in (1.3.3). Below we will refer to these factorizations as 'Haar factorizations'

Remark In chapters 8 and 9 (in the point processes context) we apply the above proposition in the situation in which $\mathbb{Y}$ is the space of 'realizations'.

There we gloss over the question of introducing the metric on such $\mathbb{Y}$. (This work has been carried out in detail in [18].)

In some cases we can apply Haar factorization to find Haar measures explicitly, as well as to obtain the criteria of existence of *bi-invariant Haar measures* (i.e. measures which satisfy both (1.3.1) and (1.3.2)).

Let $\mathbb{X}$ be a (non-commutative) group, and let $\mathbb{U}$ and $\mathbb{V}$ be two subgroups of $\mathbb{X}$. Assume that each $x \in \mathbb{X}$ admits both representations

$$\begin{aligned} x &= u_l v_r, \quad u_l \in \mathbb{U}, \quad v_r \in \mathbb{V} \\ x &= v_l u_r, \quad u_r \in \mathbb{U}, \quad v_l \in \mathbb{V} \end{aligned} \tag{1.3.4}$$

and that each of these representations is unique. (The letters 'l' and 'r' in the subscripts stand for 'left' and 'right'.)

According to (1.3.4) the *set-theoretical* product $\mathbb{U} \times \mathbb{V}$ can be mapped on $\mathbb{X}$ in two ways:

$$\begin{aligned} f_1 &: (u, v) \to uv \\ f_2 &: (u, v) \to vu \end{aligned} \tag{1.3.5}$$

and these maps are one-to-one. In other words, the product $\mathbb{U} \times \mathbb{V}$ can be used as a model for $\mathbb{X}$ *in two different ways*:

$$\begin{aligned} (u_l, v_r) &= f_1^{-1}(x) \\ (u_r, v_l) &= f_2^{-1}(x), \end{aligned}$$

where f^{-1} denotes the inverse of f.

Now

$$\begin{aligned} f_1^{-1}(ux) &= (uu_l, v_r), \quad u \in \mathbb{U}, \\ f_2^{-1}(vx) &= (u_r, vv_l), \quad v \in \mathbb{V}. \end{aligned}$$

Therefore the left-invariant Haar measure $h_{\mathbb{X}}^{(l)}$ necessarily admits two Haar factorizations, namely

$$\begin{aligned} h_{\mathbb{X}}^{(l)} &\stackrel{f_1}{=} h_{\mathbb{U}}^{(l)} \times m_1 \\ h_{\mathbb{X}}^{(l)} &\stackrel{f_2}{=} m_2 \times h_{\mathbb{V}}^{(l)}, \end{aligned} \tag{1.3.6}$$

where m_1 and m_2 are some measures on $\mathbb{V}$ and $\mathbb{U}$, respectively. The symbol $\stackrel{f}{=}$ denotes the image of the measure under the map f.

There are similar equations for the right-invariant Haar measure on $\mathbb{X}$:

$$\begin{aligned} h_{\mathbb{X}}^{(r)} &\stackrel{f_1}{=} m_1' \times h_{\mathbb{V}}^{(r)} \\ h_{\mathbb{X}}^{(r)} &\stackrel{f_2}{=} h_{\mathbb{U}}^{(r)} \times m_2', \end{aligned} \tag{1.3.7}$$

where m_1' and m_2' are some measures on $\mathbb{U}$ and $\mathbb{V}$, respectively.

In the cases where the Haar measures on the subgroups $\mathbb{U}$ and $\mathbb{V}$ are known, the partial information given by these equations can be used for the purpose of finding Haar measures on $\mathbb{X}$. Some examples are given in chapter 4.

The maps f_1 and f_2 can be used to formulate a necessary and sufficient condition of bi-invariance of $h_{\mathbb{X}}$. By repeated application of Haar factorization we conclude that any measure on $\mathbb{X}$ which is invariant with respect to the transformation

$$x \to uxv$$

is necessarily the image under f_1 of the measure

$$c \cdot h_{\mathbb{U}}^{(l)} \times h_{\mathbb{V}}^{(r)},$$

where c is a constant. Similarly, any measure on $\mathbb{X}$ which is invariant with respect to the transformations

$$x \to vxu$$

is necessarily the image of the measure

$$c \cdot h_{\mathbb{V}}^{(l)} \times h_{\mathbb{U}}^{(r)}$$

(perhaps with a different constant) under f_2. These images can be substantially different. But let us assume that *both images are proportional to a measure h on* $\mathbb{X}$. For any $A \subset \mathbb{X}$ we will have

$$h(u_2 v_1 A u_1 v_2) = h(v_1 A u_1) = h(A).$$

Since both $u_2 v_1$ and $u_1 v_2$ represent general elements from $\mathbb{X}$, this is essentially the condition defining the bi-invariant Haar measure on $\mathbb{X}$. We have come to the following result.

On $\mathbb{U} \times \mathbb{V}$ we consider two measures:

$$h_{\mathbb{U}}^{(l)} \times h_{\mathbb{V}}^{(r)} \quad \text{and} \quad h_{\mathbb{U}}^{(r)} \times h_{\mathbb{V}}^{(l)}.$$

Their respective images under f_1 and f_2 are proportional if and only if there exists a (unique) bi-invariant measure $h_{\mathbb{X}}$ on $\mathbb{X}$, $h_{\mathbb{X}}$ being proportional to the above-mentioned image measures.

The practical application of this criterion can be as follows. Each of the pairs (u_l, v_r) or (u_r, v_l) can serve as coordinates on the group $\mathbb{X}$. We express u_l and v_r in terms of u_r and v_l (both pairs of variables correspond to the same x as in (1.3.4)):

$$\begin{aligned} u_l &= \varphi_1(u_r, v_l), \\ v_r &= \varphi_2(u_r, v_l). \end{aligned} \tag{1.3.8}$$

We can assume that f_1 is trivial, i.e.

$$(u_l, v_r) = x.$$

Then f_2^{-1} is given by (1.3.8). Now application of the above criterion reduces to the usual Jacobian calculation, i.e. to a check that the transformation (1.3.8) maps $h_{\mathbb{U}}^{(r)} \times h_{\mathbb{V}}^{(l)}$ into $c \cdot h_{\mathbb{U}}^{(l)} \times h_{\mathbb{V}}^{(r)}$. In all the cases we consider in this book, bi-invariance of $h_{\mathbb{X}}$ implies that the constant c equals *one*. The typical situation will be as follows.

The elements $u \in \mathbb{U}$ and $v \in \mathbb{V}$ will depend on a finite number of parameters. Therefore c will equal the absolute value of the determinant of a matrix which we briefly denote by

$$c = \begin{vmatrix} \dfrac{\partial \varphi_1}{\partial u} & \dfrac{\partial \varphi_1}{\partial v} \\ \dfrac{\partial \varphi_2}{\partial u} & \dfrac{\partial \varphi_2}{\partial v} \end{vmatrix}.$$

Since c is a constant it is enough to calculate the value of this Jacobian at the points

$$u = 1_{\mathbb{U}} \quad \text{(the unit element of } \mathbb{U})$$

and

$$v = 1_{\mathbb{V}} \quad \text{(the unit element of } \mathbb{V}).$$

We have

$$\begin{vmatrix} \dfrac{\partial \varphi_1(u, v)}{\partial u} & \dfrac{\partial \varphi_1(u, v)}{\partial v} \\ \dfrac{\partial \varphi_2(u, v)}{\partial u} & \dfrac{\partial \varphi_2(u, v)}{\partial v} \end{vmatrix}_{\substack{u=1_{\mathbb{U}} \\ v=1_{\mathbb{V}}}} = \begin{vmatrix} \dfrac{\partial \varphi_1(u, 1_{\mathbb{V}})}{\partial u} & \dfrac{\partial \varphi_1(1_{\mathbb{U}}, v)}{\partial v} \\ \dfrac{\partial \varphi_2(u, 1_{\mathbb{V}})}{\partial u} & \dfrac{\partial \varphi_2(1_{\mathbb{U}}, v)}{\partial v} \end{vmatrix}_{\substack{u=1_{\mathbb{U}} \\ v=1_{\mathbb{V}}}}.$$

Because $\mathbb{U}$ and $\mathbb{V}$ will always be groups of transformations of the same space, from (1.3.4) we find

$$\varphi_1(u, 1_{\mathbb{V}}) \equiv u, \qquad \varphi_1(1_{\mathbb{U}}, v) \equiv 1_{\mathbb{U}},$$
$$\varphi_2(u, 1_{\mathbb{V}}) \equiv 1_{\mathbb{V}}, \quad \varphi_2(1_{\mathbb{U}}, v) \equiv v.$$

We obtain the determinant of the unit matrix, i.e. $c = 1$.

In the chapters that follow we often apply 'differential' notation for Haar measures according to table 1.3.1. Similar notation for uniquely determined invariant measures are also applied in other spaces; for instance, if $\mathscr{P}$ is the generic notation for a point in $\mathbb{R}^n$ then $d\mathscr{P}$ will denote Lebesgue measure in $\mathbb{R}^n$.

Also we use lower indexation as in (1.3.4) to avoid explicit mention of the maps f_1, f_2. Thus, $d^{(l)}u_l\, d^{(r)}v_r$ will denote a measure on $\mathbb{X}$ which is the image of the product measure $d^{(l)}u\, d^{(r)}v$ on $\mathbb{U} \times \mathbb{V}$ under the map f_1. Similarly $d^{(r)}u_r\, d^{(l)}v_l$ will denote the image of $d^{(r)}u\, d^{(l)}v$ (another measure on $\mathbb{U} \times \mathbb{V}$) under f_2. Our result for bi-invariant measures now becomes

$$dx = d^{(l)}u_l\, d^{(r)}v_r = d^{(r)}u_r\, d^{(l)}v_l. \tag{1.3.9}$$

This corresponds to writing a measure in terms of coordinates.

Table 1.3.1

Group	Element	Bi-invariant Haar	Left-invariant Haar	Right-invariant Haar
$\mathbb{X}$	$x \in \mathbb{X}$	dx	$d^{(l)}x$	$d^{(r)}x$

Remark In some cases (especially in chapter 3) we use notation like dl, dV etc. to denote infinitesimal lengths, volumes etc. The exact meaning of the notation will always be clear from the context.

1.4 Further remarks on measures

I One of the principles of the general theory of Haar measures is that on compact groups Haar measures are both finite and bi-invariant. The uniqueness up to a constant factor of course follows from the general statement quoted in §1.3. The above principle can be useful in concrete situations whenever we can point out a *finite left-invariant* (say) Haar measure h_0 (as we do in the case say, of a rotation group in §3.2). Then we automatically conclude bi-invariance and essential uniqueness of h_0. Now we show that the bi-invariance property of a finite left-invariant h_0 can be demonstrated effortlessly.

Suppose h_0 is a left-invariant Haar measure on a (compact) group $\mathbb{U}$. We take $h_0(\mathbb{U}) = 1$ for convenience. Consider the right-transformed measure $h(A) = h_0(Au)$, $A \subset \mathbb{U}$, for a fixed $u \in \mathbb{U}$. This is still a left-invariant Haar measure and obviously

$$h(\mathbb{U}) = h_0(\mathbb{U}u) = h_0(\mathbb{U}) = 1.$$

We argue by the uniqueness of left-invariant measures that

$$h = h_0.$$

This holds for all $u \in \mathbb{U}$ and so h_0 is also a right-invariant Haar.

II In chapter 4 we will use the Haar measure on the multiplicative group of positive numbers. We now denote this group by $\mathbb{X}$, $x \in \mathbb{X}$.

The map

$$y = \ln x$$

isomorphically transforms $\mathbb{X}$ into $\mathbb{T}_1$. Therefore the Haar measure on $\mathbb{X}$ is necessarily the image of the Haar (Lebesgue) measure on $\mathbb{T}_1$ under the map

$$x = \mathrm{e}^y.$$

We have

$$\mathrm{d}y = \frac{\mathrm{d}x}{x},$$

thus *the measure* dx/x *is the (unique) bi-invariant Haar measure on* $\mathbb{X}$. We also call it the 'logarithmic measure'.

The following precise result often hides behind the name of 'homothety consideration'; it will be of use in chapter 4.

Let m be a locally-finite measure on $(0, \infty)$ for which always

$$m(hB) = h^k m(B),$$

where hB denotes the image of Borel B under a homothety h (alternatively h is the corresponding rescaling factor). Then necessarily m has a density proportional to $x^{k-1}\,dx$.

Proof the measure $x^{-k}m(dx)$ is invariant with respect to homotheties and is therefore proportional to the logarithmic measure.

III In line with our concern with the question of uniqueness lies the following proposition.

Let us assume that a measure m which is defined on a product of two spaces

$$\mathbb{Y} \times \mathbb{Z}$$

has two product representations:

$$m = m_1 \times m'$$

and

$$m = m_2 \times m',$$

where m_1 and m_2 are measures on $\mathbb{Y}$ and m' is a measure on $\mathbb{Z}$. If there is a set $A_0 \subset \mathbb{Z}$ for which

$$0 < m'(A_0) < \infty$$

then the measures m_1 and m_2 are identical.

Proof For any $B \subset \mathbb{Y}$ we have

$$m(B \times A_0) = m_1(B) \cdot m'(A_0) = m_2(B) \cdot m'(A_0).$$

Therefore

$$m_1(B) = m_2(B).$$

We call the above the 'elimination of a measure factor' and use it several times in chapters 2–4.

1.5 Some topological remarks

I A number of spaces of integral geometry belong to the class of so-called fibered spaces which generalize the notion of product spaces. A space $\mathbb{X}$ is referred to as fibered when there is a map

$$\pi : \mathbb{X} \to \mathbb{Y}$$

(the projection of $\mathbb{X}$ into a space $\mathbb{Y}$) such that each fiber

$$\{x : \pi(x) = y\}$$

is homeomorphic to a space $\mathbb{Z}$ (the fiber model) which does not depend on y. Note that in §1.1, where $\mathbb{X} = \mathbb{Y} \times \mathbb{Z}$, the fibers are called *generators*. We now give an example of a fibered space which we use in §2.5.

We take the unit sphere $\mathbb{S}_2$ in $\mathbb{R}^3$. At each point $\omega \in \mathbb{S}_2$ we construct the tangent plane $t(\omega)$. Let $\mathbb{X}_1$ be the set of pairs

$$(\omega, \mathscr{P}) \quad \text{where always} \quad \mathscr{P} \in t(\omega).$$

We endow $\mathbb{X}_1$ with a topology in the following way: by definition, a sequence $\{(\omega_n, \mathscr{P}_n)\}$ converges to a point $(\omega, \mathscr{P})$ if and only if

(1) ω_n converges to ω in the usual topology on $\mathbb{S}_2$;
(2) $\mathscr{P}_n$ converges to $\mathscr{P}$ in the topology of $\mathbb{R}^3$.

The space $\mathbb{X}_1$ thus obtained is called the *tangent bundle* of $\mathbb{S}_2$; it is topologically *different* from the product $\mathbb{S}_2 \times \mathbb{R}^2$ (see [56]). The fact that it is impossible to choose coordinate systems for each tangent plane so that they vary continuously over all of the unit sphere is a simple example of the famous topological phenomenon concerning the non-existence of non-zero continuous tangent vector-fields on spheres.

Our $\mathbb{X}_1$ is a fibered space with

$$\pi(\omega, \mathscr{P}) = \omega$$

and we consider the fibers (tangent planes) as Euclidean replicas of $\mathbb{R}^2$ (i.e. we can consider congruent figures on different tangent planes).

Let us consider the planar Lebesgue measure L_2 on each $t(\omega)$. We assume that L_2 is independent of ω in the sense that congruent domains on different tangent planes have equal L_2-measures. With every measure m on $\mathbb{S}_2$ we now associate a measure μ on $\mathbb{X}_1$ by the formula

$$\mu(A) = \int L_2(A_\omega) m(\mathrm{d}\omega), \tag{1.5.1}$$

where $A_\omega = A \cap t(\omega)$ is the trace of A on $t(\omega)$. We call μ a *composition* of Lebesgue measures on fibers.

We call transformation a of $\mathbb{X}_1$ *Cavalieri* if

(1) a maps a fiber into a fiber;
(2) the image of L_2 on each fiber is again L_2 (on another fiber);
(3) a induces a map $\mathbb{Y} \to \mathbb{Y}$ of fibers which preserve m.

The non-product topology on $\mathbb{X}_1$ does not restrict the use of a type of Cavalieri principle:

On $\mathbb{X}_1$ any composition of Lebesgue measures is invariant under a Cavalieri transformation.

The proof is almost tautological.

Similar Cavalieri principles also hold for other fibered spaces in this book. We stress that in all cases the measures on fibers we compose are invariant with respect to the choice of coordinates on the fibers. As a result, their composition is uniquely determined by the measure in the space of fibers. We had this advantage in the above example.

II The remark made at the end of I concerns the different spaces of figures we consider in this book (such as spaces of lines, planes etc.). There is a general principle which governs our choice of topologies in these spaces:

They comply with the topology in the space $\mathbb{F}$ of closed sets.

By this we mean the following.

Let $\mathbb{X}$ be the basic space where our figures belong (in the case of figures which are lines, $\mathbb{X} = \mathbb{R}^2$ or $\mathbb{R}^3$). By F we now denote closed sets in $\mathbb{X} : F \in \mathbb{F}$. By definition, a sequence F_n converges in $\mathbb{F}$ if and only if it satisfies the two conditions:

(1) if an open set G hits F (i.e. if $G \cap F \neq \varnothing$) then G hits all the F_n except, at most, a finite number of them;
(2) if a compact K is disjoint of F, it is disjoint of all the F_n except, at most, of a finite number of them.

This convergence notion defines the topology on $\mathbb{F}$ (see [1]).

In many cases our figures can be considered as closed sets in $\mathbb{X}$. Then, each time, the topology on $\mathbb{F}$ induces a topology in the space of figures in question: a set A is declared open if A is an intersection of an open set in $\mathbb{F}$ with the total set of figures.

The compliance means that we will be considering homeomorphic models of the spaces of figures where the topology is induced by $\mathbb{F}$ in the above sense.

III An adequate description of a number of spaces of integral geometry requires the notion of elliptical (projective) space. We denote n-dimensional elliptical space by $\mathbf{E}_n$. A model of this space can be obtained from the unit n-dimensional sphere $\mathbb{S}_n$ by 'gluing together' every two points of $\mathbb{S}_n$ which are symmetrical (antipodal). In other words each pair of antipodal points on $\mathbb{S}_n$ is considered to be a single point of the space $\mathbf{E}_n$. Equivalently, we can take a closed half of $\mathbb{S}_n$ (a closed hemisphere) and 'glue together' the points on the boundary which are opposite to each other. Fig. 1.5.1 illustrates the latter

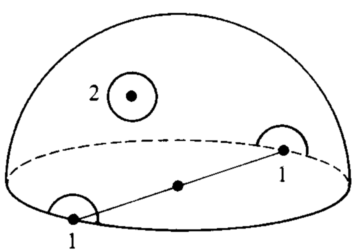

Figure 1.5.1

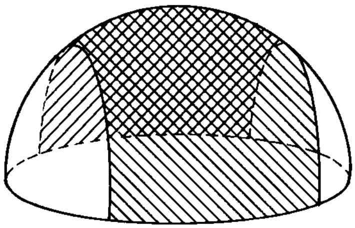

Figure 1.5.2

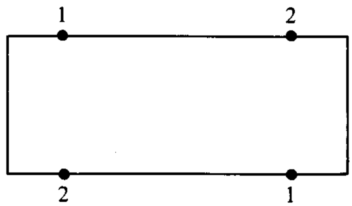

Figure 1.5.3

operation on the example of the construction of $\mathbf{E}_2$ from a two-dimensional hemisphere. The diagram shows the neighborhoods of the points on our model: the neighborhood of point 1 consists of two semicircular parts; the neighbourhood of point 2 is shown as a circle.

Consider that part of our model which is obtained by cutting off the two closed semicircular parts, as shown in fig. 1.5.2. The shaded region is homeomorphic to a rectangle with points on a pair of opposite sides glued together, as shown in fig. 1.5.3. This is the usual construction of an open Möbius band. The topology of the region that remains does not change when the region of $\mathbf{E}_2$ which is removed reduces to a point. Thus the space

$$\mathbf{E}_2 \setminus \{\text{a point}\}$$

is homeomorphic to a Möbius band.

There is a clear one-to-one map between $\mathbf{E}_2$ and spaces of such figures in $\mathbb{R}^3$ as

(1) diameters of $\mathbb{S}_2$;
(2) lines through a point O;
(3) planes through a point O (each plane of this bundle is determined by a line through O normal to the plane).

The topology of $\mathbf{E}_2$ *complies with* the closed sets' trace topology in these spaces. Therefore the spaces (1)–(3) are often described simply as $\mathbf{E}_2$.

$\mathbf{E}_1$ is obtained from $[0, \pi]$ (closed interval) by 'gluing together' its endpoints. Thus $\mathbf{E}_1$ is homeomorphic to a circle. It represents both the space of diameters of $\mathbb{S}_1$ and the space of lines through O in the plane.

1.6 Parametrization maps

Usual 'geographical' coordinates on the sphere $\mathbb{S}_2$ provide the best known example of parametrization. Actually we have a map

$$\mathbb{S}_2 \to \mathbb{S}_1 \times (0, \pi),$$

as shown in fig. 1.6.1. The image (ν, Φ) is defined for all points $\omega \in \mathbb{S}_2$ except for the 'poles' N and S.

In a typical situation a parametrization map of a space $\mathbb{X}$ onto a space $\mathbb{Y}$ will be a homeomorphism between their slitted versions

$$\mathbb{X} \backslash S_1 \to \mathbb{Y} \backslash S_2, \tag{1.6.1}$$

where the excluded sets S_1 and S_2 will be less than $\mathbb{X}$ or $\mathbb{Y}$ *in dimensionality*. As soon as such a map is specified we will write

$$\mathbb{X} \approx \mathbb{Y}. \tag{1.6.2}$$

Let m be a measure on $\mathbb{X}$. The image of m under parametrization (1.6.1) will provide an adequate description of m whenever

$$m(S_1) = 0. \tag{1.6.3}$$

Yet in general not every measure m_1 on $\mathbb{Y}$ for which

$$m_1(S_2) = 0 \tag{1.6.4}$$

can be considered to be an image of a measure on $\mathbb{X}$ (recall that in our usage measures are necessarily locally-finite).

Clearly the map converse to (1.6.1) can send a non-compact set $B \subset \mathbb{Y}$ into a subset of a compact set in $\mathbb{X}$. Therefore a measure m_1 on $\mathbb{Y}$ happens to be

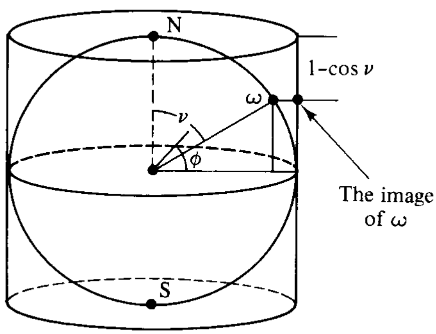

Figure 1.6.1 The circumcylinder is $\mathbb{S}_1 \times (-1, 1)$

an image of a measure on $\mathbb{X}$ whenever an additional condition

$$m_1(B) < \infty \tag{1.6.5}$$

is satisfied for every $B \subset \mathbb{Y} \backslash S_2$ with the described property.

If both (1.6.4) and (1.6.5) are satisfied, then m_1 in a sense represents a measure on $\mathbb{X}$. For instance, the measure on $\mathbb{S}_1 \times (0, \pi)$ given by the density $(v)^{-1}\, \mathrm{d}v\, \mathrm{d}\Phi$ is locally but not totally finite and fails to represent a measure on $\mathbb{S}_2$. The measure $\sin v\, \mathrm{d}v\, \mathrm{d}\Phi$ represents the area measure on $\mathbb{S}_2$.

The precautions (1.6.3)–(1.6.5) would be pointless if we could complement the map (1.6.1) by a one-to-one map between S_1 and S_2 with the property that the one-to-one map between $\mathbb{X}$ and $\mathbb{Y}$ that arises sends a compact $C \subset \mathbb{X}$ into a relatively compact set $C' \subset \mathbb{Y}$ and vice versa. Recall that a set is called relatively compact if it can be covered by a compact set.

If such a map between S_1 and S_2 can be established, then *each* measure on $\mathbb{Y}$ represents a measure on $\mathbb{X}$ and vice versa. Such a map turns $\mathbb{Y}$ into a *measure-representing model* of $\mathbb{X}$. Some examples will be given in chapter 2.

1.7 Metrics and convexity

One of the concerns of contemporary integral geometry is the interrelation between the notions of metrics, convexity and measures in the spaces of lines and planes. In this and the next section we outline the simplest facts and leave more detailed discussion of this topic for chapter 5.

Given a bounded convex domain $D \subset \mathbb{R}^2$ we define its breadth function $b(\varphi)$ to be the distance between the pair of parallel support lines of D which are orthogonal to the direction φ (see fig. 1.7.1; by definition, a support line has a point in common with ∂_D but not with the interior of D). In general $b(\varphi)$ *does not* determine a convex D in a unique way; but it does if we additionally assume that D is centrally-symmetric.

After Minkowski [30] we consider *linear continuations* b^* of the breadth

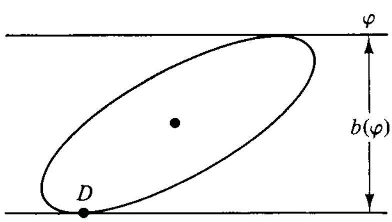

Figure 1.7.1

functions. Given a breadth function $b(\varphi)$ we put

$$b^*(\mathscr{P}) = rb(\varphi),$$

where (r, φ) are the usual polar coordinates of $\mathscr{P} \in \mathbb{R}^2$ with the origin at the symmetry center of D. There is a fundamental proposition [14]:

If $b(\varphi)$ is a breadth function of a centrally-symmetric bounded convex $D \subset \mathbb{R}^2$ which is not a line segment then

$$\rho(\mathscr{P}_1, \mathscr{P}_2) = b^*(\mathscr{P}_2 - \mathscr{P}_1)$$

is a metric in $\mathbb{R}^2$. If D is a line segment then ρ is a pseudometric.

(In this case

$$b(\varphi) = l \cdot |\cos(\varphi - \alpha)|,$$

where l is the length and α is the direction of the segment.)

We recall that a metric in $\mathbb{R}^n$ is a non-negative symmetrical function $\rho(\mathscr{P}_1, \mathscr{P}_2)$, $\mathscr{P}_1, \mathscr{P}_2 \in \mathbb{R}^n$, which satisfies the conditions

(a) $\rho(\mathscr{P}_1, \mathscr{P}_2) = 0$ if and only if $\mathscr{P}_1 = \mathscr{P}_2$;
(b) $\rho(\mathscr{P}_1, \mathscr{P}_3) \leqslant \rho(\mathscr{P}_1, \mathscr{P}_2) + \rho(\mathscr{P}_2, \mathscr{P}_3)$ for every $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$.

If ρ satisfies (b) but $\rho(\mathscr{P}_1, \mathscr{P}_2) = 0$ does not imply that $\mathscr{P}_1 = \mathscr{P}_2$ then ρ is called a *pseudometric*.

Remarkably the complete Minkowski statement also includes the inversion of the above.

If for a planar metric ρ we have

$$\rho(\mathscr{P}_1, \mathscr{P}_2) = |\mathscr{P}_1, \mathscr{P}_2| \cdot h(\varphi),$$

where $|\mathscr{P}_1, \mathscr{P}_2|$ is the Euclidean distance between $\mathscr{P}_1$ and $\mathscr{P}_2$ and the function h depends only on the direction φ from $\mathscr{P}_1$ to $\mathscr{P}_2$, then $h(\varphi)$ is the breadth function of some centrally-symmetrical convex bounded domain in $\mathbb{R}^2$ which is not a line segment. Under the same conditions any pseudometric ρ necessarily corresponds to a line segment.

Let us turn now to connections of (in general no longer translation-invariant) pseudometrics with measures in the space $\mathbb{G}$ of lines in $\mathbb{R}^2$. We describe this space in §2.2.

Let us denote by $\mathscr{P}_1|\mathscr{P}_2$ the set of lines which separate the points $\mathscr{P}_1$ and $\mathscr{P}_2$; and by $\mathscr{P}_1|\mathscr{P}_2, \mathscr{P}_3$ we denote the set of lines which separate $\mathscr{P}_1$ from $\mathscr{P}_2$ and $\mathscr{P}_3$. We have an identity which can be checked directly:

$$2I_{\mathscr{P}_1|\mathscr{P}_2\mathscr{P}_3}(g) = I_{\mathscr{P}_1|\mathscr{P}_2}(g) + I_{\mathscr{P}_1|\mathscr{P}_3}(g) - I_{\mathscr{P}_2|\mathscr{P}_3}(g),$$

where g denotes a line and I_A is the indicator function of the set A. Integration of the above with respect to any measure m in the space of lines which ascribes zero to any bundle of lines through a point yields

$$2m(\mathscr{P}_1|\mathscr{P}_2, \mathscr{P}_3) = m(\mathscr{P}_1|\mathscr{P}_2) + m(\mathscr{P}_1|\mathscr{P}_3) - m(\mathscr{P}_2|\mathscr{P}_3). \tag{1.7.1}$$

If we restrict ourselves to measures m on $\mathbb{G}$ whose values on bundles is zero, then it is quite straightforward that *each* function

$$\rho(\mathscr{P}_1, \mathscr{P}_2) = m(\mathscr{P}_1|\mathscr{P}_2) \tag{1.7.2}$$

is a linearly-additive continuous pseudometric. In particular the triangle inequality property (b) follows from (1.7.1) (where the right-hand side is non-negative).

Remarkably the following converse statement is also true.

Any pseudometric in $\mathbb{R}^2$ which is linearly-additive and continuous is generated via (1.7.2) by some measure in the space of lines, and this measure is unique.

A complete proof of this statement can be found in [3], where it was derived within the framework of combinatorial ideas (to be outlined in chapter 5). For translation-invariant cases a similar partial conclusion can be drawn using the ideas of §2.11. By Minkowski's proposition, this means that every planar symmetrical bounded convex domain is generated by a translation-invariant measure in the space $\mathbb{G}$.

Which of the above notions virtually generalize to many dimensions, in particular to $\mathbb{R}^3$?

The significance of the breadth functions $b(\omega)$ for a complete description of centrally-symmetrical convex domains survives together with Minkowski's propositions. (In $\mathbb{R}^3$, $b(\omega)$ equals the distance between parallel support planes of a convex D which are orthogonal to the spatial direction ω.) Also the principle (1.7.2) that *measures generate metrics* remains true. (In $\mathbb{R}^3$ we have to interprete $\mathscr{P}_1|\mathscr{P}_2$ as the set of planes separating $\mathscr{P}_1$ from $\mathscr{P}_2$; m becomes a measure in $\mathbb{E}$, the space of planes in $\mathbb{R}^3$.)

Yet in $\mathbb{R}^3$ the situation with the *inversion* of the latter principle changes: in $\mathbb{R}^3$ there exist linearly-additive, continuous metrics which do not admit the representation (1.7.2) with any measure m on $\mathbb{E}$. Accordingly the bounded symmetrical convex domains (bodies) in $\mathbb{R}^3$ split into two subclasses: *zonoids*, i.e. those which correspond to metrics generated by measures in $\mathbb{E}$, and those which do not. (We dwell upon these questions later in §§2.12, 5.10, 6.1 and 6.2.)

Breadth functions are useful in the study of projections of convex bodies on planes (of course the projections are planar convex domains).

Let Ω be the direction normal to the plane on which we project a convex body $D \subset \mathbb{R}^3$, and let $b(\omega)$ be the breadth function of D. By $\langle\Omega\rangle$ we denote the circle of directions orthogonal to Ω (they lie in the plane of projection). The *breadth function $b(\varphi)$ of the projection coincides with the restriction of $b(\omega)$ to the set $\langle\Omega\rangle$.* This is clearly demonstrated by fig. 1.7.2.

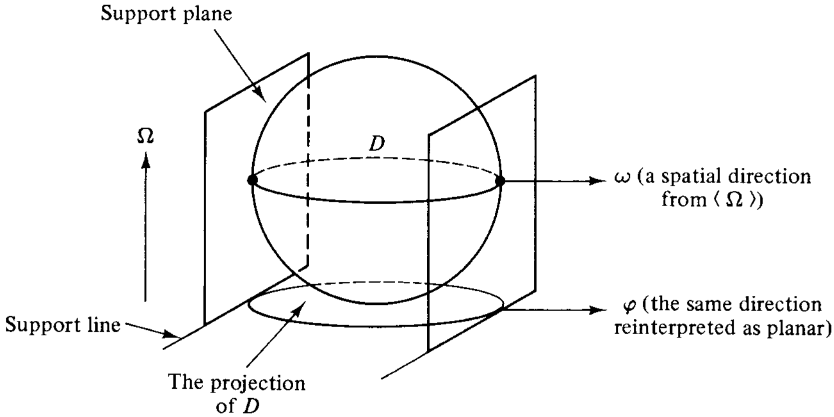

Figure 1.7.2

1.8 Versions of Crofton's theorem

In books on integral geometry (including [1] and [2]) the following problem is discussed in detail. Given two non-intersecting planar convex domains D_1 and D_2, find the *invariant measure* of the set of lines separating D_1 from D_2. By invariant measure we understand the unique (up to a constant factor) measure in the space of lines in the plane which is invariant with respect to Euclidean motions. We discuss this measure in detail in chapter 3, and it reappears frequently in other chapters.

The solution attributed to Crofton [2] is that the value in question equals 'the least length of a closed string drawn round D_1 and D_2 and crossing over itself at a point O, minus the lengths of the perimeters of D_1 and D_2'. (See fig. 1.8.1.)

Let us consider a version of this result in which D_1 and D_2 are replaced by line segments.

On the plane we have two line segments, δ_1 and δ_2, situated as shown in fig. 1.8.2.

The invariant measure of the lines which hit both δ_1 and δ_2 (or, equivalently, separate s_1 from s_2) equals

$$|d_1| + |d_2| - |s_1| - |s_2|, \tag{1.8.1}$$

where $|d|$ stands for the length of d.

In fact, versions of these simple results for non-invariant measures in the space of lines in the plane lie at the source of the theory of combinatorial integral geometry ([3]).

Although we outline the theory later on (in chapter 5), we will need the following simple fact in chapter 2.

Let us denote by $[\delta]$ the set of lines which hit the segment δ. Except for the lines passing through the endpoints of δ_1 and δ_2 we have (see fig. 1.8.2)

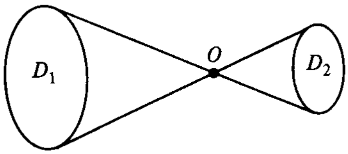

Figure 1.8.1

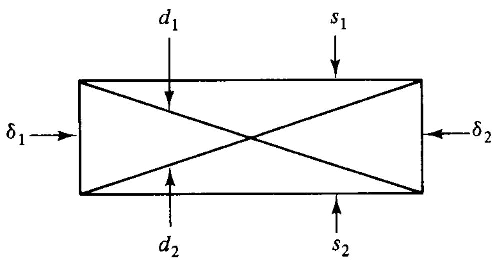

Figure 1.8.2 We denote by d_1 and d_2 the *diagonals* and by s_1 and s_2 the *sides* of the quadrilateral

$$2I_{[\delta_1]\cap[\delta_2]}(g) = I_{[d_1]}(g) + I_{[d_2]}(g) - I_{[s_1]}(g) - I_{[s_2]}(g), \tag{1.8.2}$$

where g denotes a line and I_A is the indicator function of the set A. (To get the proof it is enough to consider four different positions of g). Integration of (1.8.2) with respect to any measure m in the space of lines (which ascribes zero to any bundle of lines through a point) yields

$$2m([\delta_1] \cap [\delta_2]) = m([d_1]) + m([d_2]) - m([s_1]) - m([s_2]). \tag{1.8.3}$$

Of course (1.7.1) can be considered as a special case of this relation when δ_1 and δ_2 are situated so as to form two sides of a triangle.