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An Introduction to Independence for Analysts

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PREFACE

The purpose of this book is to explain what it means for a proposition to be independent of set theory, and to describe how independence results can be proved by the technique of forcing. We do this by presenting an application of forcing to a deep and interesting problem in analysis. Our application is, by current standards in set theory, fairly non-technical, and so it offers an excellent setting in which to exhibit to analysts these new techniques from set theory.

Most analysts will have a certain acquaintance with logic and set theory. They will know naïve set theory up to the level of ordinals and cardinals. They will have heard that forcing is a powerful technique that enables one to prove that certain propositions of set theory are independent of specified axioms, and, in particular, that Cohen developed the method of forcing in his proof that the Continuum Hypothesis (CH) is independent of the basic axioms of set theory, ZFC. They may also know of more recently formulated axioms, such as Martin's Axiom (MA), which can be used to establish independence results without the necessity of knowing any of the technicalities of forcing.

However, it is possible that analysts harbour two negative feelings about these matters. First, they may feel that, although logic and set theory are of interest in their own right, they have little to contribute concerning questions which "really" arise in mathematical analysis, and so can be safely left to their disciples. But this is not true. For example, several natural questions about sets of real numbers cannot be resolved in the theory ZFC. For some of these questions the set-theoretic entanglements are quite subtle, in that these questions *can* be resolved by invoking the existence of large cardinal numbers. Here is a specific example. It is not difficult to show that, if $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f(B)$ is Lebesgue measurable for each Borel subset B of $\mathbb{R}$. Now suppose that $f,g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Is $f(\mathbb{R}\setminus g(B))$ Lebesgue measurable for each Borel set B? This problem cannot be decided in ZFC. However if there is a measurable cardinal, then these sets are indeed all Lebesgue measurable.

The main example that we present in this work did arise naturally in analysis. It concerns the automatic continuity of homomorphisms from a Banach algebra of continuous functions into an arbitrary Banach algebra, and the formulation of the problem was such that a solution was expected (perhaps naïvely) in naïve set theory. But eventually it was discovered that the existence of discontinuous homomorphisms from the algebras under consideration could not be decided in ZFC.

Secondly, analysts may feel that the technicalities of forcing are too arcane to be readily accessible. We seek to challenge this view by presenting a reasonably complete account of forcing which is comprehensible to non-logicians: we hope that it will bring them to the point at which they can appreciate the application, which we shall give in detail, of these new ideas to the above automatic continuity problem. We should say, however, that it is not our intention to teach the practical use of forcing in general, but rather to explain quite explicitly how the method of forcing does yield independence and consistency results, and to exemplify this by the study of our chosen example.

Thus this book is directed towards non-logicians, and in particular to analysts. We shall give an account of the background in logic that we shall require, and we shall explain the key notions of proof, of consistency, and of independence. The approach to independence will be through the theory of models: Gödel's completeness theorem allows us to recast independence questions as problems of the construction of models with certain properties.

We shall give full proofs of the results about forcing and Martin's Axiom that we shall need. En route to our main theorem, we shall prove that CH is independent from ZFC, a result that we believe should be known to all mathematicians.

We also hope that our account will be useful for students of set theory as a preliminary to other works. Very little knowledge of analysis is required to follow the details of our example, and the background that is required is given in Chapter 1.

We now describe the problem in analysis that we are to consider. Let X be a compact Hausdorff space, and let $C(X,\mathbb{C})$ be the set of all continuous, complex-valued functions on X. The set $C(X,\mathbb{C})$ is a commutative algebra with respect to the pointwise operations. Set

$$|f|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C(X,\mathbb{C})).$$

Then $|\cdot|_X$ is an algebra norm on $C(X,\mathbb{C})$ - it is the <u>uniform norm</u> - and $(C(X,\mathbb{C}), |\cdot|_X)$ is a Banach algebra. The question we ask is whether or not each algebra norm on $C(X,\mathbb{C})$ is necessarily equivalent to the uniform norm: if so, the topological structure of $C(X,\mathbb{C})$ as a normed algebra would be completely determined by its algebraic structure. This question was first discussed by Kaplansky in 1948, and it was eventually "resolved" independently by Dales and by Esterle in 1976: if the Continuum Hypothesis holds, then, for each infinite compact (Hausdorff) space X, there is an algebra norm on $C(X,\mathbb{C})$ which is not equivalent to the uniform norm.

The appeal to CH in this theorem was thought at the time (at least by those involved) to be an accident and a weakness of the given proofs. However, also in 1976, it was proved by Solovay, using a condition of Woodin, that, if there is a model of set theory, then there is a model of set theory in which each algebra norm on each $C(X,\mathbb{C})$ is equivalent to the uniform norm. Shortly afterwards a different, and easier, approach to the theorem was developed by Woodin; the approach used some techniques of Kunen. It is this approach, which involves Martin's Axiom, that we present here.

These results are included in Woodin's thesis, written at the University of California, Berkeley, and a presentation for experts in forcing is given in the recent volume [45] of Jech, but apart from this ours is the first account of the theorem to appear in print. The original work of Solovay has never been published. A revised version of Woodin's thesis is to appear [72]; this memoir, which will also include several more complicated independence results, is written for logicians, and we can say with quiet confidence that it will be unintelligible to analysts.

Chapter 1 contains only analysis: following the seminal work of Bade and Curtis, we shall analyze the structure of an arbitrary homomorphism from $C(X,\mathbb{C})$ into a Banach algebra. In Chapter 2, we shall discuss partially ordered sets, Boolean algebras, and ultraproducts, topics which form the background to much of our later work, and in Chapter 3 we shall relate our question to one which is amenable to the techniques which are given to us in the theory of forcing: if there is an algebra norm on any $C(X,\mathbb{C})$ which is not equivalent to the uniform norm, then there is a free ultrafilter $\mathcal{V}$ on $\mathbb{N}$ and an isotonic map from a subset $\langle Z\rangle/\mathcal{V}$ of the ultraproduct $(\mathbb{R}^{\mathbb{N}}/\mathcal{V},<_{\mathcal{V}})$ into $(\mathbb{N},<_F)$, where $<_F$ is the Fréchet order on $\mathbb{N}$. Also in Chapter 3, we shall discuss the structure of the algebras $\ell^{\infty}/\mathcal{U}$ and $c_o/\mathcal{U}$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$: we formulate some apparently interesting open questions about these algebras.

In Chapter 4, we shall explain how independence can be established by the construction of models. In this chapter, we shall list and discuss the axioms of set theory, and we shall formulate the axiom NDH: "For each compact space X, each homomorphism from $C(X,\mathbb{C})$ into a Banach algebra is continuous.". It is a theorem of Kaplansky, which we discuss in Chapter 1, that for each $C(X,\mathbb{C})$ the uniform norm is the minimal possible algebra norm. Thus the axiom NDH holds if and only if each algebra norm on each $C(X,\mathbb{C})$ is equivalent to the uniform norm.

Chapter 5 first recalls the elementary theory of ordinals and cardinals, and then discusses Martin's Axiom: we give some applications of this axiom to analysis. In Chapter 6, we shall give a final reduction of our problem: we shall prove in ZFC + MA + ¬CH that, if NDH fails, then there is an embedding of a certain totally ordered set $(\mathfrak{R},<)$ into $(\mathbb{N},<_F)$.

In Chapter 7, we shall give our development of forcing: we shall work with Boolean-valued models and with Boolean-valued universes. In our treatment, we place more emphasis than is customary on explaining exactly how forcing arguments yield consistency results. (Let us note for the logician that our treatment is a little non-standard, in that we do not discuss standard models.) At this stage, we prove that CH is independent of ZFC.

Having learnt what forcing is, we must learn how to iterate the process: we study iterated forcing in Chapter 8, essentially proving that "a forcing extension of a forcing extension is a forcing extension". Here again, we are more explicit and give more details than is usual, but we do this at little cost in length. Finally we can conclude the proof of the main theorem of the book: if ZFC is consistent then so is ZFC + MA + NDH.

The practical significance of our main theorem for an analyst is that it is futile to attempt to prove that there is a compact space X and an algebra norm on $C(X,\mathbb{C})$ which is not equivalent to the uniform norm.

We conclude this preface with some remarks about our style of writing.

Throughout, we shall expand points that we guess will be less familiar to analysts (this is a dangerous game for us), and we shall try to use notation which is closer to that traditional in analysis than to that preferred by logicians. But this is not always sensible: logicians do, on occasion, adopt their notation for good reasons, related to the nature of their subject. We have had to find an amicable compromise. For example, let $f : S \to T$ be a function taking the value $f(s)$ at $s \in S$. Now analysts are usually happy to write $f(A)$ for the range of the restriction of f to a subset A of S. But this is dangerously ambiguous, for it may be that S is a transitive set, so that A is also an element of S. In this case, we have followed the analysts, trusting to context to arbitrate the ambiguity.

It is often necessary to be more precise in logic than one would be in analysis. For example, a Boolean algebra is strictly a sextuple

$$\mathfrak{B} = \langle B, \wedge, \vee, ', 0, 1 \rangle$$

satisfying certain rules: outside logic, one usually identifies $\mathfrak{B}$ with its underlying set B, and we shall do this in Chapters 1 - 6 of this book. However, in Chapters 7 and 8 it seems to be necessary for us to maintain the distinction between $\mathfrak{B}$ and B.

Here are some particular notations that we shall use. (An index of notation is given on pages 235-236.) First, $\mathbb{N} = \{1,2,3,\ldots\}$, whereas $\mathbb{Z}^+ = \{0,1,2,\ldots\}$ (so that, again, we are following the analysts). Second, $S \subset T$ allows the possibility that $S = T$. Third, in a phrase such as

$$\text{rad } A = \{a \in A : e - ab \in \text{Inv } A \quad (b \in A)\}$$

the bracket "$(b \in A)$" means "for all b in A".

We give some references in the notes at the end of the chapters to the theorems proved. Although some results are new or have simplified proofs, it is of course not the case that unattributed results are claimed as original.

The authors first discussed this work when the first author visited Caltech in 1984: we are grateful to Professor W. Luxemburg for arranging this visit. We also acknowledge the award of a Visiting Fellowship to the second author by the United Kingdom Science and Engineering Research Council that enabled him to visit Leeds in September, 1984. The second author is an A.P. Sloan Fellow, and we are grateful that this fellowship provided financial support for us to meet at later times, both in the snows of Leeds and in the sunshine of Pacific Palisades, and for us to arrange the typing of this book.

A number of people, both analysts and set theorists, have read some chapters of a preliminary version of this work, and we are very grateful to them for their comments. They include William Bade, John Derrick, Peter Dixon, Peter McClure, Neil Mowbray, John Truss and Tom Ransford. We give special thanks to K.P. Hart and to the referee for reading the entire book in manuscript and for making many valuable suggestions: of course, the decisions on the approach, the (possibly tendentious) opinions, and the errors are all our responsibility.

We owe a special debt of thanks to our typist, Mrs. Joan Bunn. A large amount of information in a work of this type is carried by the details of the symbolism and the fonts. We have tried hard to eliminate errors in this area (presumably unsuccessfully): without the skill and accuracy of Mrs. Bunn's transcription, the process of correcting our errors would surely have diverged.

H.G. Dales and W.H. Woodin,
Leeds, April, 1987