1 HOMOMORPHISMS FROM ALGEBRAS OF CONTINUOUS FUNCTIONS

Throughout this book, we shall work in the naïve set theory familiar to analysts: the formalization of this theory is the axiom system ZFC, which will be discussed in Chapter 4.

We first summarize some elementary facts about Banach algebras that we shall use. The account in the standard text of Rudin [58, Chapters 10 and 11] covers essentially all that we shall require.

All the algebras that we shall consider are linear and associative, and their underlying field is either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. An algebra is unital if it has an identity; in this case, we often denote the identity by e. The algebra formed by adjoining an identity to a non-unital algebra A is denoted by $A^{\#}$, and we take $A^{\#} = A$ if A is unital.

The set of invertible elements of a unital algebra A is denoted by $\operatorname{Inv} A$.

A character on a complex algebra A is a non-zero homomorphism from A onto $\mathbb{C}$. The set of characters on A is the character space of A, written Φ_A.

Let A be a commutative algebra. An ideal I of A is modular if the quotient algebra A/I is unital. The (Jacobson) radical, written $\operatorname{rad} A$, of A is defined to be the intersection of the maximal modular ideals of A; if A has no such ideals, then $\operatorname{rad} A = A$, and in this case A is a radical algebra. (So our convention is that a radical algebra is necessarily commutative.) The algebra A is semisimple if $\operatorname{rad} A = \{0\}$.

It is standard that

$$\text{rad } A = \{a \in A : e - ab \in \text{Inv } A^{\#} \quad (b \in A^{\#})\},$$

where e is the identity of $A^{\#}$. Several times we shall use the fact that, if $a \in A$ and $ab = a$ for some $b \in \text{rad } A$, then $a = 0$.

An element a of an algebra A is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. We write nil A for the set of nilpotent elements in A. If A is commutative, then nil A is an ideal in A (it is the nilradical of A), and nil $A \subset$ rad A.

An ideal P in a commutative algebra A is a prime ideal if either $a \in P$ or $b \in P$ whenever $a,b \in A$ with $ab \in P$. The algebra A is an integral domain if the zero ideal is a prime ideal in A. We shall occasionally use the standard algebraic fact that, if I is an ideal in A and if $a \in A$ is such that $a^n \notin I$ $(n \in \mathbb{N})$, then there is a prime ideal P with $I \subset P$ and $a \notin P$.

1.1 DEFINITION

Let A be an algebra over a field k. A seminorm on A is a map $p : A \to \mathbb{R}$ such that:

(i) $p(a) \geqslant 0 \quad (a \in A)$;

(ii) $p(\alpha a) = |\alpha| p(a) \quad (\alpha \in k, a \in A)$;

(iii) $p(a + b) \leqslant p(a) + p(b) \quad (a,b \in A)$;

(iv) $p(ab) \leqslant p(a)p(b) \quad (a,b \in A)$.

A norm on A is a seminorm p such that

(v) $p(a) \neq 0 \quad (a \in A \backslash \{0\})$.

The algebra A is seminormable (respectively, normable) if there is a non-zero seminorm (respectively, a norm) on A.

It would be more precise to say "algebra seminorm" and "algebra norm", but this extra precision seems to be unnecessary for us.

A norm is usually denoted by $\|\cdot\|$. A Banach algebra is an algebra with a complete norm. Each normed algebra A is a dense, normed subalgebra of a Banach algebra, the completion of A ([8, 1.12]).

The starting point for the "automatic continuity" theory of Banach algebras is the following basic fact. Let A be a Banach algebra, and let $\phi \in \Phi_A$. Then ϕ is continuous, and in fact

$$\|\phi\| \leqslant 1.$$

The character space Φ_A is a locally compact space with respect to the relative weak *-topology from the dual space of A. (We adopt throughout the convention that a locally compact space is Hausdorff.) If A is a commutative, unital Banach algebra, then Φ_A is compact and non-empty.

Let A be a commutative Banach algebra. If $\Phi_A \neq \emptyset$, then the map $\phi \mapsto \ker \phi$ is a bijection from Φ_A onto the set of maximal modular ideals of A, and so

$$\text{rad } A = \bigcap\{\ker \phi : \phi \in \Phi_A\}. \qquad (1)$$

If $\Phi_A = \emptyset$, then A is a radical algebra. Since each character is continuous on A, each maximal modular ideal is closed, and so $\text{rad } A$ is a closed ideal in A.

Let A be an algebra, and let $a \in A$. Then $\sigma(a)$, the spectrum of a, is the set

$$\sigma(a) = \{z \in \mathbb{C} : ze - a \notin \text{Inv } A^{\#}\}.$$

Let A be a Banach algebra. A fundamental theorem ([58, 10.13]) asserts that $\sigma(a)$ is a non-empty, compact subset of $\mathbb{C}$ for each $a \in A$. Set

$$r(a) = \sup\{|z| : z \in \sigma(a)\} \quad (a \in A).$$

Then $r(a)$ is the spectral radius of a, and the spectral radius formula asserts that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf \|a^n\|^{1/n} \quad (a \in A).$$

If A is commutative, then $\sigma(a) = \{\phi(a) : \phi \in \Phi_{A^\#}\}$, and so $a \in \text{rad}\, A$ if and only if $r(a) = 0$ ([58, 11.9]). Thus a commutative Banach algebra is a radical algebra if and only if $\|a^n\|^{1/n} \to 0$ as $n \to \infty$ for each $a \in A$.

We now describe in more detail the specific Banach algebras that we are studying.

Let X be a topological space. Then $C(X,\mathbb{C})$ (respectively, $C(X)$) denotes the set of continuous, complex-valued (respectively, real-valued) functions on X. (In making this definition, we implicitly assume that X is non-empty.) Then $C(X,\mathbb{C})$ and $C(X)$ are, respectively, complex and real algebras with respect to the pointwise algebraic operations on X. The constant function 1 is the identity of these algebras.

Let S be a subset of X. For each bounded function f on S, set

$$|f|_S = \sup\{|f(x)| : x \in S\}.$$

Then $|\cdot|_S$ is the <u>uniform norm on</u> S.

Let X be a compact (Hausdorff) space. Then $(C(X,\mathbb{C}), |\cdot|_X)$ is a commutative, unital Banach algebra. For $x \in X$, set $\varepsilon_x(f) = f(x)$ $(f \in C(X,\mathbb{C}))$. Then it is a standard result ([58, 11.13(a)]) that the map $x \mapsto \varepsilon_x$ is a homeomorphism of X onto $\Phi_{C(X,\mathbb{C})}$, and we shall henceforth identify these two spaces.

Let $f \in C(X,\mathbb{C})$, and set $\bar{f}(x) = \overline{f(x)}$ $(x \in X)$, where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. Then $\bar{f} \in C(X,\mathbb{C})$. A subalgebra A of $C(X,\mathbb{C})$ is <u>self-adjoint</u> if $\bar{f} \in A$ $(f \in A)$. We shall use the standard fact ([58, 11.18]) that each closed, self-adjoint subalgebra of $C(X,\mathbb{C})$ which contains 1 is isometrically isomorphic to $C(Y,\mathbb{C})$ for a certain compact space Y.

Let $f \in C(X,\mathbb{C})$. Then $Z(f) = f^{-1}(\{0\})$ is the <u>zero-set of</u> f. A subset F of X is a <u>zero-set</u> if

$F = Z(f)$ for some $f \in C(X,\mathbb{C})$.

Let I be an ideal in $C(X,\mathbb{C})$. The <u>hull</u> of I is the closed subset of X:

$$h(I) = \bigcap\{Z(f) : f \in I\}.$$

Let F be a closed set in X. Then we throughout set

$$I(F) = \{f \in C(X,\mathbb{C}) : Z(f) \supset F\}$$

and

$$J(F) = \{f \in C(X,\mathbb{C}) : Z(f) \text{ is a neighbourhood of } F\}.$$

It is easily seen that $I(F)$ and $J(F)$ are, respectively, the maximal and minimal ideals in $C(X,\mathbb{C})$ whose hull is F, that $I(F)$ is closed, and that $J(F)$ is dense in $I(F)$. Let $x \in X$. Then we write J_x and M_x for $J(\{x\})$ and $I(\{x\})$, respectively. We see that $\{M_x : x \in X\}$ is the family of maximal ideals in $C(X,\mathbb{C})$.

We use the same notations $I(F)$, $J(F)$, J_x and M_x for the analogous ideals in $C(X)$ when we are considering real-valued functions.

We write $\ell^\infty(\mathbb{C})$ (respectively, ℓ^∞) for the set of bounded, complex-valued (respectively, real-valued) sequences on $\mathbb{N}$, and we set

$$|\alpha|_{\mathbb{N}} = \sup\{|\alpha_n| : n \in \mathbb{N}\} \quad (\alpha = (\alpha_n) \in \ell^\infty(\mathbb{C})).$$

Then $(\ell^\infty(\mathbb{C}), |\cdot|_{\mathbb{N}})$ is a complex, commutative, unital Banach algebra. Temporarily, we denote its character space by Φ. For $n \in \mathbb{N}$, set $\varepsilon_n(\alpha) = \alpha_n$ $(\alpha = (\alpha_n) \in \ell^\infty(\mathbb{C}))$, and, for $\alpha \in \ell^\infty(\mathbb{C})$, set $\hat{\alpha}(\phi) = \phi(\alpha)$ $(\phi \in \Phi)$. The following results are standard and are easily proved; they follow from the Gelfand-Naimark theorem ([58, 11.18]), for example, and are given in greater generality in [35, 8.3].

(i) The map $n \mapsto \varepsilon_n$, $\mathbb{N} \to \Phi$, is a homeomorphism with dense range.

(ii) The map $\alpha \mapsto \hat{\alpha}$, $\ell^\infty(\mathbb{C}) \to C(\Phi,\mathbb{C})$, is an isometric isomorphism.

Thus the compact space Φ has the properties of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. In our approach, we define $\beta\mathbb{N}$ to be Φ: different, topological characterizations are given in [36] and [69], for example, and we shall give a characterization in terms of ultra-filters in Chapter 2. Henceforth, we identify $\mathbb{N}$ with a subset of $\beta\mathbb{N}$ and $\ell^\infty(\mathbb{C})$ with $C(\beta\mathbb{N},\mathbb{C})$. For example, if $\mathfrak{p} \in \beta\mathbb{N}$, then $M_\mathfrak{p}$ is a maximal ideal in $\ell^\infty(\mathbb{C})$.

Let σ be a subset of $\mathbb{N}$, and let χ_σ be the characteristic function of σ. Clearly $\{\mathfrak{p} \in \beta\mathbb{N} : \hat{\chi}_\sigma(\mathfrak{p}) = 1\} = \bar{\sigma}$, the closure of σ in $\beta\mathbb{N}$. Thus, if $\{\sigma,\tau\}$ is a partition of $\mathbb{N}$, then $\{\bar{\sigma},\bar{\tau}\}$ is a partition of $\beta\mathbb{N}$ into clopen sets: we shall use this fact several times. It implies that $\overline{\sigma \cap \tau} = \bar{\sigma} \cap \bar{\tau}$ for $\sigma,\tau \subset \mathbb{N}$, and that, for $\mathfrak{p} \in \beta\mathbb{N}$, $\{\bar{\sigma} : \sigma \subset \mathbb{N}, \mathfrak{p} \in \bar{\sigma}\}$ is a base of neighbourhoods of $\mathfrak{p}$ in $\beta\mathbb{N}$ consisting of clopen sets. The existence of this base shows that $\beta\mathbb{N}$ is a totally disconnected space.

We note that, if $\mathfrak{p} \in \beta\mathbb{N}$, then $J_\mathfrak{p}$ is a prime ideal in $\ell^\infty(\mathbb{C})$. For take $f,g \in \ell^\infty(\mathbb{C})$ with $fg \in J_\mathfrak{p}$. Then

$$\mathfrak{p} \in \{n : (fg)(n) = 0\}^-$$

$$= \{n : f(n) = 0\}^- \cup \{n : g(n) = 0\}^-,$$

and so either $\mathfrak{p} \in \{n : f(n) = 0\}^-$, in which case $f \in J_\mathfrak{p}$, or $\mathfrak{p} \in \{n : g(n) = 0\}^-$, in which case $g \in J_\mathfrak{p}$. Thus $J_\mathfrak{p}$ is prime.

We write $c_0(\mathbb{C})$ (respectively, c_0) for the set of complex-valued (respectively, real-valued) sequences which converge to zero. Then $c_0(\mathbb{C})$ is an ideal in $\ell^\infty(\mathbb{C})$, and our identification equates it with the ideal $I(\beta\mathbb{N}\setminus\mathbb{N})$ in $C(\beta\mathbb{N},\mathbb{C})$.

We write $c_{00}(\mathbb{C})$ and c_{00} for the ideals in $c_0(\mathbb{C})$ and c_0, respectively, consisting of sequences (α_n) such that $\alpha_n = 0$ for all but finitely many n.

The question that we shall study in this book is the following.

QUESTION

Let X *be a compact space. Is each norm on the algebra* $C(X,\mathbb{C})$ *necessarily equivalent to the uniform norm* $|\cdot|_X$*?*

The question was raised by Kaplansky in lectures around 1948, and it was Kaplansky who gave the first partial result ([48]).

1.2 THEOREM

Let X be a compact space, and let $\|\cdot\|$ be a norm on $C(X,\mathbb{C})$. Then $|f|_X \leqslant \|f\|$ $(f \in C(X,\mathbb{C}))$.

<u>Proof</u>

Let A be the completion of the normed algebra $(C(X,\mathbb{C}), \|\cdot\|)$. Then A is a commutative, unital Banach algebra.

The map $\phi \mapsto \phi|C(X,\mathbb{C})$, $\Phi_A \to \Phi_{C(X,\mathbb{C})}$, is a continuous injection, and so we may regard Φ_A as a closed subspace of X. Suppose, if possible, that $\Phi_A \neq X$, and take $x \in X\backslash\Phi_A$. Then there exist f and g in $C(X,\mathbb{C})$ such that $f(\Phi_A) = g(\Phi_A) = \{0\}$, such that $g(x) = 1$, and such that $fg = g$. By (1), $f \in \text{rad } A$, and so $g = 0$, a contradiction. Thus $\Phi_A = X$.

Let $f \in C(X,\mathbb{C})$, and let $x \in X$. Since $x \in \Phi_A$, $|f(x)| \leqslant \|f\|$, and so $|f|_X \leqslant \|f\|$, as required. ∎

It follows immediately from 1.2 (via the open mapping theorem) that each <u>complete</u> norm on $C(X,\mathbb{C})$ is equivalent to the uniform norm, and so there is an inequivalent norm on $C(X,\mathbb{C})$ if and only if there is an incomplete norm.

Let $\|\cdot\|$ be an incomplete norm on $C(X,\mathbb{C})$, and let A be the completion of $(C(X,\mathbb{C}), \|\cdot\|)$. Then the embedding $C(X,\mathbb{C}) \to A$ is a discontinuous monomorphism. On the other hand, suppose that θ is a discontinuous homomorphism from $C(X,\mathbb{C})$ into a Banach algebra. Set

$$\|f\| = \max\{|f|_X, \|\theta(f)\|\} \quad (f \in C(X,\mathbb{C})).$$

Then $\|.\|$ is an incomplete norm on $C(X,\mathbb{C})$. Thus our question is equivalent to the question of the existence of a discontinuous homomorphism from $C(X,\mathbb{C})$ into a Banach algebra, and it will usually be convenient to formulate it in this way.

The next advance in the study of norms on $C(X,\mathbb{C})$ after the work of Kaplansky was the seminal paper of Bade and Curtis of 1960 ([2]). In this paper, the structure of an arbitrary homomorphism from $C(X,\mathbb{C})$ into a Banach algebra was analysed, and the key notions of singularity set and radical homomorphism (see below) were introduced. Later, in 1976, Johnson ([47]) extended Bade and Curtis's theorem, and used the extension to show that, if there is an incomplete norm on $C(X,\mathbb{C})$ for any compact space X, then there exists $\mathfrak{p} \in \beta\mathbb{N}\backslash\mathbb{N}$, a radical Banach algebra R, and a non-zero homomorphism θ from $c_o(\mathbb{C})$ into R such that $\ker \theta \supset J_{\mathfrak{p}}$. It is this reduction theorem that we shall need in Chapter 3. Our proof is slightly more direct, and gives a little more information, than that obtained by following the original route.

The main technical device in the proof of Bade and Curtis's theorem is the following main boundedness theorem, which originates in [2, Theorem 2.1].

1.3 THEOREM

Let A and B be Banach algebras, and let θ be a homomorphism from A into B. Suppose that there are sequences (a_n) and (b_n) in A such that $a_m b_n = 0$ $(m,n \in \mathbb{N},\ m \neq n)$. Then there exists a constant C such that

$$\|\theta(a_n b_n)\| \leqslant C\,\|a_n\|\,\|b_n\| \quad (n \in \mathbb{N}).$$

Proof

We can suppose that $\|a_n\| = \|b_n\| = 1 \quad (n \in \mathbb{N})$.

Suppose, if possible, that the result is false. Then for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, we may choose $n(i,j) \in \mathbb{N}$ such that the map $(i,j) \mapsto n(i,j)$ is injective and such that

$$\|\theta(u_{ij}v_{ij})\| \geq 4^{i+j} \quad (i,j \in \mathbb{N}),$$

where $u_{ij} = a_{n(i,j)}$ and $v_{ij} = b_{n(i,j)}$. Set

$$f_i = \sum_{\ell=1}^{\infty} v_{i\ell}/2^{\ell} \quad (i \in \mathbb{N}),$$

so that each $f_i \in A$. Choose $j(i) \in \mathbb{N}$ so that

$$\|\theta(f_i)\| \leq 2^{j(i)} \quad (i \in \mathbb{N}),$$

and set

$$g = \sum_{k=1}^{\infty} u_{k,j(k)}/2^k,$$

so that $g \in A$. Now, for $i \in \mathbb{N}$,

$$gf_i = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} u_{k,j(k)}v_{i\ell}/2^{k+\ell}$$

$$= u_{i,j(i)}v_{i,j(i)}/2^{i+j(i)}$$

and so

$$\|\theta(gf_i)\| \geq 4^{i+j(i)}/2^{i+j(i)} = 2^{i+j(i)}.$$

But

$$\|\theta(gf_i)\| \leq \|\theta(g)\|\,\|\theta(f_i)\| \leq 2^{j(i)}\,\|\theta(g)\|,$$

and so $\|\theta(g)\| \geq 2^i$ for all $i \in \mathbb{N}$, a contradiction.

Thus the result holds. ∎

We introduce some further notation. Let X be a compact space. We write N_x for the family of open neighbourhoods of a point x of X. If U is an open subset of X, then $K_U = I(X\backslash U)$, so that K_U consists of those functions of $C(X,\mathbb{C})$ which vanish off U, and K_U is a closed ideal in $C(X,\mathbb{C})$. For example, $K_\emptyset = \{0\}$.

1.4 DEFINITION

Let θ be a homomorphism from $C(X,\mathbb{C})$ into a Banach algebra. A point $x \in X$ is a singularity point for θ if, for each $U \in N_x$, $\theta|K_U$ is discontinuous. The set of singularity points is the singularity set for θ.

1.5 DEFINITION

A radical homomorphism at x is a non-zero homomorphism from a maximal ideal M_x of $C(X,\mathbb{C})$ into the radical of a commutative Banach algebra.

Let $x \in X$, and let $\theta : M_x \to A$ be a homomorphism into a commutative Banach algebra A. We *claim* that θ is a radical homomorphism if and only if $\theta|J_x = 0$. For suppose that $\theta(M_x) \subset \text{rad } A$, and take $f \in J_x$. Then there exists $g \in M_x$ with $fg = f$, and $\theta(f)\theta(g) = \theta(f)$, so that $\theta(f) = 0$. Thus $\theta|J_x = 0$. Conversely, suppose that $\theta|J_x = 0$, and take $\phi \in \Phi_A$. Then $\phi\circ\theta \in \Phi_{M_x} \cup \{0\}$, and $(\phi\circ\theta)|J_x = 0$. Since J_x is dense in M_x, $\phi\circ\theta = 0$, and so $\theta(M_x) \subset \ker \phi$. Thus $\theta(M_x) \subset \text{rad } A$, and θ is a radical homomorphism. The claim is established.

It is further clear that there is a radical homomorphism at x if and only if M_x/J_x is seminormable.

We shall several times use the elementary fact that, if F is an infinite subset of a regular space X, then there exist $x_n \in F$ and $U_n \in N_{x_n}$ for $n \in \mathbb{N}$ such that $U_m \cap U_n = \emptyset$ $(m \neq n)$. For take $x,y \in F$ with $x \neq y$. Then there exist $U \in N_x$ and $V \in N_y$ with $x \notin \bar{V}$, $y \notin \bar{U}$, and $U \cup V = X$. Either $U \cap F$ or $V \cap F$ is infinite, and so we can choose $x_1 \in F$ and an open set W_1 such that