ON THE COMPUTATIONAL COMPLEXITY OF RATIONAL HOMOTOPY

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"[A]lthough the procedures developed for [computing homotopy groups] are finite, they are much too complicated to be considered practical."

E. Brown, 1956 [3]

The research summarized here may be viewed as a vindication of Brown's remarks. Given a finite simply-connected CW complex X, a common problem in algebraic topology is to evaluate $\pi_n(X)$. A much simpler problem is to determine the rank of $\pi_n(X)$. The latter problem is shown to belong to the class of #P-hard problems, which are believed to require more than polynomial time to compute deterministically. Computing the Hilbert series of a graded algebra or the Poincaré series of a local Artinian ring is also #P-hard.

Full details will be given in a forthcoming paper [2]. This note contains a brief overview of the results.

SUMMARY OF RESULTS

In computer science, a "problem" is a function f from a subset of $\mathbb{N}$, the non-negative integers, to $\mathbb{N}$. Sometimes the argument of this function, also called the "input", may be viewed naturally as a single integer, while at other times it may encode, through some fixed injection $\eta: \coprod_{j=1}^{\infty} \mathbb{Z}^j \to \mathbb{N}$, the finite description of some other object. (The map η is said to <u>$\mathbb{N}$-encode</u> the finite description.) Regardless of the proper interpretation of the input, a computer scientist may imagine that a machine or algorithm exists which can accept an arbitrary $N \in \mathcal{D}om(f)$ as input and after $\tau(N)$ steps deliver $f(N)$ as output. Given f, he or she may seek theoretical lower bounds on the function $\tau(N)$ or seek algorithms (machines) on which $\tau(N)$ exhibits a certain level of efficiency.

It is not at all obvious how one gives an efficient finite description of an arbitrary finite simply-connected CW complex X. Simplicial and semi-simplicial descriptions of X tend to be "too large."

If we restrict our attention to rational homotopy, however, Quillen's minimal Lie algebra model provides a complete description of the rational homotopy type of X, a description whose size is roughly comparable with the number of cells of X and with the complexity of the attaching maps. The problem "compute rank $(\pi_n(X))$" is therefore interpreted as follows: given an integer n and the Quillen model for a space X, both $\mathbb{N}$-encoded, determine $\dim_{\mathbb{Q}}(\pi_n(X)\otimes\mathbb{Q})$.

Related problems are to be viewed similarly. For example, "Compute the Hilbert series of a finitely presented connected graded algebra" means: given an $\mathbb{N}$-encoded list of generators and relations for a graded $\mathbb{Q}$-algebra $A = \bigoplus_{m=0}^{\infty} A_m$ and an integer n, determine $\dim_{\mathbb{Q}}(A_n)$.

Computer scientists have developed a scale or continuum along which various problems may be placed according to their complexity. Certain classes of problems, including the classes known as $\mathcal{P}$, #$\mathcal{P}$-complete, and "computable in exponential time," serve as landmarks. In particular, there is a transitive, reflexive relation on the set of problems, called "Turing reducible in polynomial time" and denoted $\leq_T^P$, by which f_2 is at least as hard as f_1 if $f_1 \leq_T^P f_2$. The problems f_1 and f_2 are "Turing equivalent", denoted $f_1 \approx_T f_2$, if $f_1 \leq_T^P f_2$ and $f_2 \leq_T^P f_1$. The goal of this research is to locate the problem "compute rational homotopy" and some related problems upon this continuum.

Our results are summarized in Figure 1. More difficult problems are placed higher on the scale, and Turing equivalent problems have been bracketed. Problems at a specific difficulty level are marked by a bar crossing the vertical axis. Classes of problems which encompass a range along the continuum are labeled at the upper or lower end of their range.

The purpose of Figure 1 is to give a visual overview of our results, and some technical points were sacrificed for crispness. For instance, one might conclude from the picture that any problem unsolvable in exponential time is #$\mathcal{P}$-hard, but this has not been proved (and is probably false). Nor have we proved that the three Turing equivalence classes marked by bars crossing the axis must actually be distinct.

As to interpretation, bear in mind that the class #$\mathcal{P}$-hard starts near the bottom of Figure 1 even though this class is viewed by computer scientists as being "very difficult". In other words, except for

"$\mathcal{P}$", the section of the scale shown in Figure 1 actually starts far above such familiar computer mainstays as "solve a linear system of equations", "find the roots of a polynomial", or "factor the integer N."

There are no known algorithms which can evaluate a #$\mathcal{P}$-hard problem in less than exponential time. Algorithms which require exponential time are generally thought of as being beyond the scope of today's machinery to implement efficiently. Thus computing rational homotopy groups and the other problems listed in Figure 1 may truly be described as very complex problems.

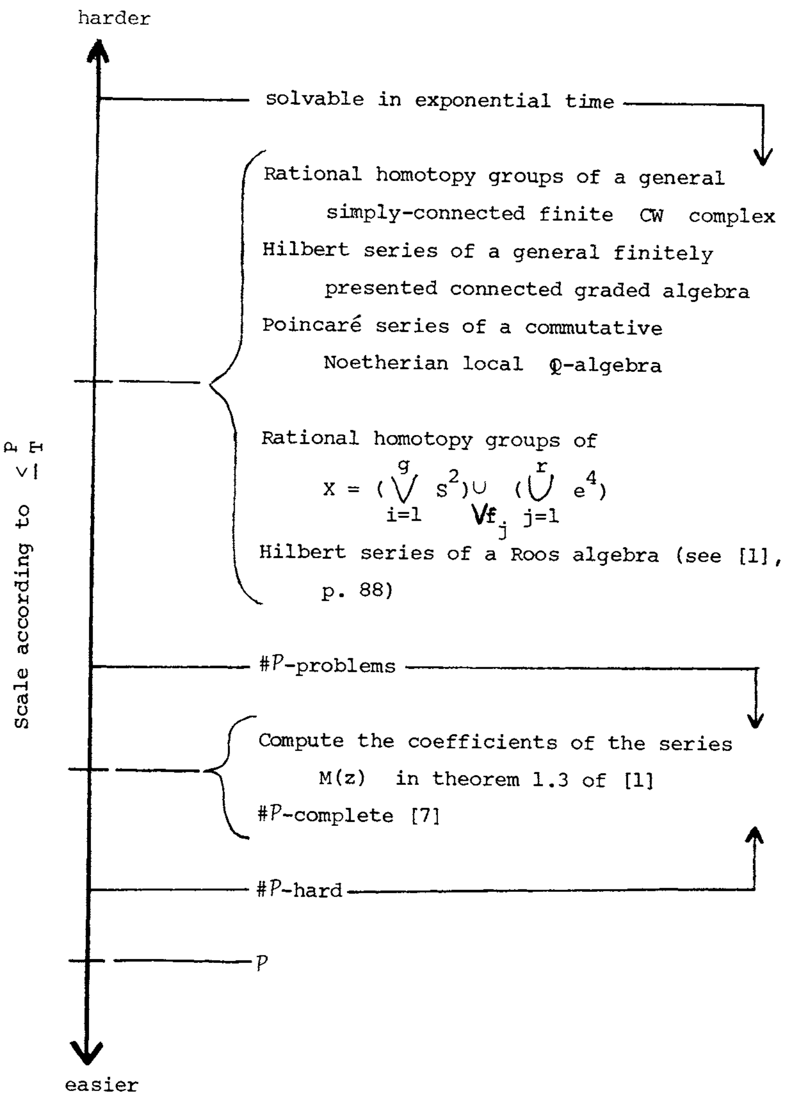
harder
solvable in exponential time
Rational homotopy groups of a general simply-connected finite CW complex
Hilbert series of a general finitely presented connected graded algebra
Poincaré series of a commutative Noetherian local Q-algebra
Rational homotopy groups of
$X = (\bigvee_{i=1}^{g} S^2) \cup_{\vee f_j} (\bigcup_{j=1}^{r} e^4)$
Hilbert series of a Roos algebra (see [1], p. 88)
#P-problems
Compute the coefficients of the series M(z) in theorem 1.3 of [1]
#P-complete [7]
#P-hard
P
easier
Scale according to $\leq_P^T$

FIGURE 1

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SEGAL'S BURNSIDE RING CONJECTURE AND THE HOMOTOPY LIMIT PROBLEM

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This is a somewhat informal discussion of Segal's Burnside ring conjecture. We give an outline of how it is proved, and in later sections discuss how it can be applied to other problems of identification of a "homotopy fixed point set." We outline the paper. §1 is a discussion of Atiyah's theorem on the K-theory of classifying spaces of finite groups, which is the first theorem of this type. §2 discusses Dyer Lashof maps and the connection of $B\Sigma_n$ with $\Omega^{\infty}S^{\infty}$, and thereby motivates Segal's conjecture. In §3, we define and outline some of the properties of equivariant stable homotopy theory, which is the key piece of machinery in stating the conjecture in its correct generality, and is in fact the key ingredient in the reduction from the general p-group case to the elementary Abelian case. In §4, we outline the work of Lin (1980) and Adams et al. (1985), who prove the elementary Abelian case. In §5, we give an account of the results of my paper (Carlsson, 1984), in which the reduction to the elementary Abelian case is accomplished. In §6, we state the so-called "homotopy limit" problem, of which Segal's conjecture is a special case. Other cases include Sullivan's conjecture and the conjectured existence of a "descent spectral sequence" for algebraic K-theory. Finally, in §7, we show how Segal's conjecture may be applied to study these problems.

As stated above, the discussion is informal, and for complete statements and proofs of all theorems, one should refer to the original sources. Technical points are sometimes suppressed in order to continue the flow of the discussion; for instance, $\lim^1$-arguments are omitted.

A few words about references are in order. Atiyah's theorem can be found in Atiyah's original paper; a different (and perhaps more enlightening) proof of a more general result may be found in Atiyah and Segal (1969).

For the results of §2, the Dyer-Lashof maps and the connections with $B\Sigma_n$ are treated well in Madsen and Milgram (1979): more thorough treatments include the books of Adams (1978) and May (1972). The Burnside ring may be found in Tom Dieck (1979), as well as in the work of Dress (1969). Equivariant stable homotopy was defined by Segal in Segal (1970), and was presumably the motivation for the original conjecture. It is also discussed in Tom Dieck's book, and in Adams (1984). A more encyclopedic account due to Lewis, May, McClure, and Steinberger will appear in due course (Lewis et al., to appear). The references for the elementary Abelian case are Lin (1980), Gunawardena (1980), and Adams et al. (1985). The primary reference for §5 is Carlsson (1984). The Adams spectral sequence is discussed in Adams (1974) and Switzer (1975). The nerve construction for posets and categories is available in Quillen (1973), and results of the paper by Quillen (1978) shed much light on the reduction procedure. For the necessary results on group cohomology, we refer to Quillen and Venkov (1972). To my knowledge, the homotopy limit problem, as such, was formulated by Thomason in (1983). Sullivan's conjecture was posed in Sullivan (1970), and its proof in the case of trivial G-action by H. R. Miller appears in Miller (1974). For a discussion of the descent problem in algebraic K-theory, and its solution for "Bott periodic" K-theory, see Thomason's paper (to appear). More general discussions of algebraic K-theory appear in Quillen (1970,1973). The theorem of Suslin appears in Suslin (1983). Bousfield and Kan (1972) is an excellent source for cosimplicial spaces, and in particular for the cosimplicial space of a triple.

The author wishes to apologize for omissions, made in the interest of brevity. Segal and Stretch (1981) and Laitinen (1979) made earlier contributions to the study of the Segal conjecture for abelian groups. We have also omitted recent work on Sullivan's conjecture by Jackowski, as well as applications of Miller's result by McGibbon-Neisendorfer and Zabrodsky.

Finally, the author wishes to express his thanks to many people for valuable discussions on many of the subjects discussed in this paper. An incomplete list includes: J. F. Adams, E. Friedlander, M. J. Hopkins, W. C. Hsiang, M. Karoubi, I. Madsen, M. E. Mahowald, R. J. Milgram, H. R. Miller, V. P. Snaith, R. W. Thomason, and C. Weibel.

1 ATIYAH'S THEOREM

In 1960, Atiyah (1961) gave a complete description of the group $[BG, BU \times \mathbb{Z}]$ of homotopy classes of maps from BG to BU, where BG and BU denote the classifying spaces of a finite group G and the infinite unitary group, respectively. (The $\mathbb{Z}$ factor denotes $\mathbb{Z}$ viewed as a discrete space.) Since homotopy classification of maps in general is a very complicated problem, and classifying spaces of finite groups can be very complicated, the simplicity of the answer is striking, and we describe it.

We recall first that $BU \times \mathbb{Z}$ admits two pairings, $\oplus : (BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z}$ and $\otimes : (BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z}$, arising from Whitney sum and tensor product of vector bundles respectively, which satisfy the usual associativity and distributivity relations. Accordingly, $[BG, BU \times \mathbb{Z}]$ becomes a ring, and we will describe it in terms of a classical algebraic invariant of G, the complex representation ring. Recall how it is defined. Let Rep(G) denote the set of isomorphism classes of unitary representations of G ; equivalently, Rep(G) is just the set of complex characters of G. Rep(G) becomes an Abelian monoid under the direct sum operation, and a semiring under direct sum and tensor product. After adjoining formal additive inverses, we obtain the complex representation ring of G, R[G]. Classically, it was known as the character ring. It is a free, finitely generated Abelian group whose rank is equal to the number of conjugacy classes of elements of G. R[G] is contrafunctorial in G ; moreover, $R[\{e\}] \cong \mathbb{Z}$, so the inclusion $\{e\} \to G$ induces a ring homomorphism $\epsilon : R[G] \to \mathbb{Z}$, called the augmentation. The kernel of ϵ, I[G], is called the *augmentation ideal* in R[G].

Let ρ be a unitary representation of G ; thus, ρ is a homomorphism from G to U(n) for some n. The functoriality of the classifying space construction means that we have a map $f_\rho : BG \xrightarrow{B\rho} BU(n) \xrightarrow{Bi} BU \times \mathbb{Z}$ where $i : U(n) \to U$ is the inclusion. It is elementary to check that f_ρ depends only on the isomorphism class of ρ, and that we therefore obtain a function $Rep[G] \to [BG, BU \times \mathbb{Z}]$. One checks that this induces a homomorphism of rings $R[G] \to [BG, BU \times \mathbb{Z}]$. It is tempting to guess that this map is an isomorphism; it isn't quite, but we may describe precisely what the necessary modifications to R[G] are. Let R be a commutative ring, and $I \subseteq R$ an ideal. Then by the I-adic completion of R, $\hat{R}_I$, we mean the inverse limit $\varprojlim_n R/I^n$. Atiyah then proved that there is a factorization

$$\begin{array}{ccc} R[G] & \rightarrow & \hat{R}[G]_{I[G]} \\ & \searrow & \downarrow \\ & & [BG,BU] \end{array}$$

where the right hand vertical arrow is an isomorphism. This gives a completely algebraic description of the ring [BG, BU]. The reader may wonder, however, whether the ring $\hat{R}[G] = \hat{R}[G]_{I[G]}$ is a particularly tractable algebraic invariant. If G is a p-group, then $\hat{R}[G]$ may be described by the Cartesian square of rings.

$$\begin{array}{ccc} \hat{R}[G] & \rightarrow & R[G] \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \\ \downarrow & & \downarrow \ \epsilon \otimes \mathrm{id} \\ \mathbb{Z} & \rightarrow & \hat{\mathbb{Z}}_p \end{array} .$$

In the general (composite order) case, the map $R[G] \rightarrow \hat{R}[G]$ may not be injective, but $\hat{R}[G]$ may be described by induction theorems. In short, $\hat{R}[G]$ is quite a tractable invariant.

2 DYER LASHOF MAPS, THE BURNSIDE RING, AND SEGAL'S CONJECTURE

We let $Q(S^0) = \varinjlim_n \Omega^n S^n$, where the maps in the directed system are induced by suspension. For a point $x \in \mathbb{R}^n$, and a number r, we let $f_{x,r} : B_r(x) \rightarrow \mathbb{R}^n \cup \{\infty\}$ be defined by $f_{x,r}(z) = \frac{z-x}{r-|z-x|}$. For any finite subset $S = \{x_1, \ldots, x_n\}$, and collection of radii $R = (r_1, \ldots, r_k)$, for which $B_{r_i}(x_i) \cap B_{r_j}(x_j) = 0$, we define

$$\begin{aligned} f_{S,R}(z) &= f_{x_i, r_i}(z) \quad \text{for } z \in B_{r_i}(x_i) \\ &= \infty \quad \text{if } z \notin \bigcup_i B_{r_i}(x_i) . \end{aligned}$$

Let $C_{n,k} = \{(x_1,\ldots,x_k, r_1,\ldots,r_k) | x_i \in \mathbb{R}^n, B_{r_i}(x_i) \cap B_{r_j}(x_j) = 0\}$. Then Σ_k act on $C_{n,k}$ by permutation of coordinates, and the action is clearly free. Moreover, the construction above gives a map $C_{n,k}/\Sigma_k \rightarrow (\Omega^n S^n)_k$, the component of $\Omega^n S^n$ of maps of degree k. Passing to the direct limit gives a map from $\lim_{\vec{n}} C_{n,k}/\Sigma_k \to (Q(S^o))_k \cdot \lim_{\vec{n}} C_{n,k}$ can be shown to be contractible, so the domain is in fact a model for $B\Sigma_k$. After translating by a map of degree $-k$, we get map $B\Sigma_k \xrightarrow{D_k} Q(S^o)$, which are homotopy-compatible with the inclusions $B\Sigma_k \to B\Sigma_{k+1}$. These are the Dyer-Lashof maps, which produce a map $B\Sigma_\infty \xrightarrow{D} Q(S^o)_0$.

Theorem (Barratt, Priddy, Quillen, see Priddy, 1971). *The map D induces an isomorphism on integral homology.*

Of course, the map is not a homotopy equivalence, since $Q(S^O)_0$ is not a $K(\Sigma_\infty, 1)$-space, but it suggests that the relationship between the symmetric groups and stable homotopy theory should be a strong one. In particular, we could ask a question quite analogous to the one studied by Atiyah, with $Q(S^O) \times \mathbb{Z}$ replacing $BU \times \mathbb{Z}$.

Question: *Can $[BG, Q(S^O) \times \mathbb{Z}]$ be described in purely algebraic terms, where G is a finite group?*

We first note that if $\rho : G \to \Sigma_k$ is any homomorphism, we may form the composite

$$f_\rho : BG \xrightarrow{B\rho} B\Sigma_k \xrightarrow{D_k} Q(S^o),$$

and obtain an element in $[BG, Q(S^O)]$. One is led to conjecture, by analogy with Atiyah's theorem, if in an appropriate sense all elements of $[BG, Q(S^O)]$ are obtained in this way. To understand what the "appropriate sense" is, we must first understand what the analogue of $R[G]$ is.

By a finite G set, we simply mean a finite set X together with an action of G; equivalently, we mean a homomorphism from G to $\Sigma(X)$, the group of all permutations of X. There is an evident notion of isomorphism of G-sets acting on a fixed set X, and the isomorphism classes correspond to conjugacy classes of homomorphisms $G \to \Sigma(X)$. If X_1 and X_2 are two finite G-set then the disjoint union and products of X_1 and X_2, $X_1 \coprod X_2$ and $X_1 \times X_2$, are also finite G-sets in the evident way. This fact means that the collection of isomorphism classes of finite G-sets form a commutative monoid under disjoint union, and a commutative semiring under disjoint union and products. After adjoining formal additive inverses, we actually obtain a commutative ring $A(G)$, called the *Burnside ring* of G.