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1. BASIC TERMINOLOGY

1. Basic graph-theoretic terms

In this section we shall define some basic graph-theoretic terms that will be used in this book. Other graph-theoretic terms which are not included in this section will be defined when they are needed.

Unless stated otherwise, all graphs are finite, undirected, simple and loopless. A directed graph is called a digraph and a directed edge is called an arc. A multigraph permits more than one edge joining two of its vertices. The number of edges joining two vertices u and v is called the multiplicity of uv and is denoted by $\mu(u,v)$.

The cardinality of a set S is denoted by $|S|$. Let $G = (V,E)$ be a graph where $V = V(G)$ is its vertex set and $E = E(G)$ is its edge set. The order (resp. size) of G is $|V|$ (resp. $|E|$) and is denoted by $|G|$ (resp. $e(G)$). Two vertices u and v of G are said to be adjacent if $uv \in E$. If $e = uv \in E$, then we say that u and v are the end-vertices of e and that the edge e is incident with u and v. Two edges e and f of G are said to be adjacent if they have one common end-vertex. If $uv \in E$, then we say that v is a neighbour of u. The set of all neighbours of u is called the neighbourhood of u and is denoted by $N_G(u)$ or simply by $N(u)$ if there is no danger of confusion. The valency (or degree) of a vertex u is $|N(u)|$ and is denoted by $d(u)$. The maximum (resp. minimum) of the valencies of the vertices of G is called the maximum (resp. mininum) valency of G and is denoted by $\Delta(G)$ (resp. $\delta(G)$).

A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G such that whenever $u, v \in V(H)$ are adjacent in G then they are also adjacent in H is called an induced subgraph of G. An induced subgraph of G having vertex set (or a subgraph of G induced by) $\{v_1, \ldots, v_k\}$ is denoted by $\langle v_1, \ldots, v_k \rangle$. The subgraph induced by $V(G) - \{v_1, \ldots, v_k\}$ is denoted by $G - \{v_1, \ldots, v_k\}$ or by $G - v_1 - \ldots - v_k$.

A vertex of valency 0 is called an isolated vertex. If all the vertices of G have the same valency, d say, then we say that G is regular of degree d and we write deg G = d. A regular graph of degree 3 is called a cubic graph. If G is a regular graph of order n such that deg G = 0 (resp. n - 1), then G is called a null graph (resp. complete graph) and is denoted by O_n (resp. K_n). If the vertex set of G can be partitioned into two sets V_1 and V_2 such that every edge of G joins one vertex in V_1 to one vertex in V_2, then G is called a bipartite graph. A bipartite graph having bipartition V_1 and V_2 is said to be complete if each vertex in V_1 is adjacent to every vertex in V_2. The complete bipartite graph having bipartition V_1 and V_2 such that $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$. A complete bipartite graph $K_{1,r}$ is called a star and is denoted by S_{r+1}. The Petersen graph G(5,2) is a cubic graph having vertex set $V = \{u_0, \ldots, u_4, v_0, \ldots, v_4\}$ and edge set $E = \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+2}) \mid i = 0, \ldots, 4\}$ where all the subscripts are taken modulo 5. The generalized Petersen graph G(n,k) ($n \geq 5$, $0 < k < n$) is the cubic graph having vertex set $\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}$ and edge set $\{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+k}) \mid i = 0, \ldots, n - 1\}$ where all the subscripts are taken modulo n. The graphs whose vertices and edges are the vertices and edges of the five regular solids are called the platonic graphs.

An independent set of edges, or matching, in G is a set of edges no two of which are adjacent. A matching in G that includes every vertex of G is called a 1-factor in G.

A sequence of distinct edges of the form $v_0 v_1, v_1 v_2, \ldots, v_{r-1} v_r$ is called a path of length r from v_0 to v_r. If the vertices $v_0, v_1, \ldots, v_r$ are all distinct, then the path is called a chain (or open chain), whereas if the vertices are all distinct except that $v_r = v_0$, then the path is a cycle (or circuit). The length of a shortest open chain from a vertex u to a vertex $v \neq u$ is called the distance between u and v and is denoted by $\partial(u,v)$. The maximum distance between two vertices of G is called the diameter of G and is denoted by d(G). The length of a shortest cycle in G is called the girth of G and is denoted by $\gamma(G)$. The length of a longest cycle in G is called the circumference of G. A cycle of length n is denoted by C_n and a shortest path (open chain) of length n is denoted by P_n. If G has an open chain P that includes every vertex of G, then P is called a Hamilton path (or H-path) of G. A cycle that includes all the vertices of G is called a Hamilton cycle of G. If G has a Hamilton cycle, then G is said to be Hamiltonian.

Two graphs G and H are said to be disjoint if they have no vertex in common. Suppose G and H are two disjoint graphs. Then the (disjoint) union $G \cup H$ of G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ and the join $G + H$ of G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. A graph H is said to be obtained from a graph G by inserting a vertex $w (\notin V(G))$ into an edge uv of G if $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) - \{uv\}) \cup \{uw, wv\}$. Two graphs H_1 and H_2 are said to be homeomorphic if both of them can be obtained from the same graph G by inserting vertices into the edges of G. The complement $\bar{G}$ of a graph G is the graph having vertex set V(G) such that two vertices in $\bar{G}$ are adjacent if and only if they are not adjacent in G. The line graph L(G) of a graph G is the graph having vertex set E(G) such that two vertices in L(G) are adjacent if and only if their corresponding edges in G are adjacent.

A connected graph is a graph such that any two vertices are connected by a path. A graph G which is not connected is the (disjoint) union of some connected subgraphs which are called the components of G. A component of a graph is odd if it has an odd number of vertices. The number of odd-components of G is denoted by o(G). A vertex v of G is a cut-vertex if $G - v$ has more components than that of G. Analogous to the cut-vertex is the concept of a bridge. A bridge of a graph G is an edge e such that the graph $G - e$ obtained from G by deleting the edge e has more components than that of G.

A (proper) vertex-colouring of G is a map $\pi : V(G) \to \{1,2,\ldots\}$ such that no two adjacent vertices have the same image. The chromatic number $\chi(G)$ of G is the minimum cardinality of all possible images of vertex-colourings of G.

The following two theorems will be applied :

Dirac's theorem *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

Tutte's theorem A graph G has a 1-factor if and only if

$$o(G - S) \leq |S| \quad \text{for all } S \subseteq V(G).$$

2. Groups acting on sets

In this section we shall define some basic graph-theorectic terms and state some theorems on group theory that will be used in this book.

Suppose X is a nonempty set with (or without) a structure. Then the set of all structure-preserving permutations of the elements of X forms a group under composition of maps. For instance, if G is a graph and X is the vertex set of G, then the set of all permutations of X preserving the adjacency of vertices forms a group, called the automophism group of G.

Historically, the theory of groups dealt at first with such permutation groups and later dealt with only abstract groups. However, it has been found that the notion of group actions (or groups acting) on sets, which passes an abstract group to a concrete permutation group, provides good counting techniques. As a result, the notion of group actions on sets plays an important role in the theory of finite groups.

We say that a group (an abstract group) G acts on a nonempty set X if to each g in G and each x in X there corresponds a unique element g(x) in X such that for every $x \in X$ and for every $g, h \in G$, $gh(x) = g(h(x))$ and $1(x) = x$, where 1 is the identity element in G.

Now suppose G acts on a set $X \neq \phi$. Then to each g in G, there corresponds a permutation ϕ_g in Σ_X, the set of all permutations of X, given by $\phi_g : x \to g(x)$. It is clear that $\phi : G \to \Sigma_X$ given by $\phi : g \to \phi_g$ is a (group) homomorphism. We call ϕ the permutation representation of G corresponding to the group action.

Conversely, suppose $\phi : G \to \Sigma_X$ is a homomorphism. Then G acts on X when we define $g(x) = \phi(g)(x)$ for each $g \in G$ and each $x \in X$. Thus a group action of G on X can be defined alternatively as a homomorphism from G to Σ_X.

From the second definition, we can see that the notion of a group acting on a set $X \neq \phi$ is more general than that of a permutation group on X, because in the former case unequal group elements can give rise to equal permutations, i.e. the map $\phi : g \to \phi_g$ need not be one-to-one. If the map $\phi : g \to \phi_g$ is one-to-one, then G is said to act faithfully on X.

Suppose G acts on a set $X \neq \phi$. It is not difficult to show that if we define a relation ~ on X by setting $x_1 \sim x_2$ if there exists g in G such that $g(x_1) = x_2$, then ~ is an equivalence relation on X. Hence, for each x in X, we can define the G-orbit of x, denoted by Orb(x), to be the set $\{g(x) \mid g \in G\}$ and the stabilizer G_x (or Stab(x)) of x in G to be the set $\{g \in G \mid g(x) = x\}$. It is not difficult to show that G_x is a subgroup of G and that $|Orb(x)| = [G : G_x]$ where $[G : G_x]$ is the index of G_x in G. Thus if G is finite, then $|Orb(x)| = |G|/|G_x|$.

Let G act on a set $X \neq \phi$. The action is said to be transitive if it has just one orbit; otherwise it is intransitive. An action of G on X is doubly transitive if for any two ordered pairs (x_1,x_2), (y_1,y_2) of distinct elements of X, there is some g in G such that $g(x_i) = y_i$, $i = 1, 2$. An action of G on X is said to be regular if it is transitive and $G_x = \{1\}$ for each x in X. Hence a regular action is faithful. The following theorem will be used in the study of vertex-transitive graphs in Chapter 3.

Theorem 2.1 If a finite group G acts transitively on X, then for any $x \in X$, $|X| = |G|/|G_x|$.

If G acts on X and x, y ε X are such that $g(x) = y$, then it is not difficult to show that $G_y = g^{-1}G_x g$. This fact can be used to prove the following theorem.

Theorem 2.2 (Burnside's counting theorem) If a finite group G acts on $X \neq \phi$, then the number of orbits of G is

$$\frac{1}{|G|} \sum_{g \in G} \psi(g)$$

where $\psi(g) = |\{x \in X \mid g(x) = x\}|$.

We can define an action of G on itself by conjugation : for each g, $x \in G$, we write $x^g = g^{-1} xg$. Then $Orb(x) = \{g^{-1}xg \mid g \in G\}$ is the conjugacy class of x in G and $Stab(x) = \{g \in G \mid g^{-1} xg = x\} = \{g \in G \mid$

$xg = gx\} = C_G(x)$ is the centralizer of x in G. Hence, if G is a finite group having k distinct conjugacy classes, then from the fact that $|Orb(x)| = [G : G_x]$, we have the class equation of G :

$$|G| = \sum_{i=1}^{k} |Orb(x_i)| = \sum_{i=1}^{k} [G : C_G(x_i)] \quad (1)$$

where $x_1, \ldots, x_k$ are the representatives of the k conjugacy classes.

Let Z(G) be the set of elements x in G such that $C_G(x) = G$. Then Z(G) is the centre of G, and from (1) we have

$$|G| = |Z(G)| + \sum [G : C_G(y_i)] \quad (2)$$

where y_i runs through a set of representatives of the conjugacy classes which contain more than one element.

Suppose G is a finite group of order $p^k m$ where p is a prime and $p \nmid m$ (p does not divide m). Then a subgroup H of G such that $|H| = p^k$ is called a Sylow p-subgroup of G.

Using (2), H. Wielandt produced a very short proof of the following theorem which we shall apply in Chapter 3.

Theorem 2.3 (Sylow's theorem) Suppose G is a finite group of order $p^k m$ where p is a prime and $p \nmid m$. Then

(i) G contains a subgroup of order p^i for every $i \leq k$.

(ii) Any two Sylow subgroups of G are conjugate in G, i.e. if H_1 and H_2 are Sylow p-subgroups, then there exists $g \in G$ such that $H_2 = g^{-1} H_1 g$.

(iii) The number of Sylow p-subgroups of G is a divisor of m and is congruent to 1 modulo p.

(iv) Any subgroup of order p^i, $i \leq k$, is contained in a Sylow p-subgroup.

(For a 2-page proof of this theorem, see N. Jacobson: Basic Algebra 1, pp. 78-79.)

Suppose G acts transitively on X. For each subset Y of X and each g in G, let $g(Y) = \{g(y) \mid y \in Y\}$. A subset Y of X is said to be a block for the action if for each g in G, either $g(Y) = Y$ or $g(Y) \cap Y = \phi$. It is clear that ϕ, X and all the 1-element subsets of X are blocks for the action. These blocks are called the trivial blocks. The action is said to be primitive if the only blocks are the trivial blocks; otherwise the action is imprimitive.

For results concerning primitive group actions, the readers can refer to N. L. Biggs and A. T. White : Permutation Groups and Combinatorial Structures, London Mathematical Society Lecture Note Series 33, 1979; or J. S. Rose : A Course on Group Theory, Cambridge University Press, 1978.

We shall apply Polya's Pattern Counting Theorem in Chapter 3. Before we state Polya's theorem, we first define some terms.

Let D and R be nonempty sets. Let ψ be a map with domain D and range R and let G be a permutation group on D. We define a binary relation ~ on the set R^D of all maps from D to R as follows : $\psi_1 \sim \psi_2$ if there exists $g \in G$ such that

$$\psi_1(d) = \psi_2(g(d)) \quad \text{for every } d \in D \tag{3}$$

It can be shown that this binary relation ~ is an equivalence relation on R^D. The equivalence classes determined by ~ are called the patterns. The patterns correspond to the distinct ways of distributing $|R|$ objects into $|D|$ cells when equivalence between ways of distribution is introduced by the group acting on D.

To each element r in R (called the store), we assign a weight w(r) which is an element in a commutative ring. The inventory of R is defined to be $\Sigma_{r \in R}\, w(r)$. Now for each $\psi \in R^D$, we define the weight $W(\psi)$ of ψ to be $\Pi_{d \in D}\, w(\psi(d))$, and for each $S \subseteq R^D$ we define the inventory of S to be $\Sigma_{\psi \in S}\, W(\psi)$. From the definitions, it follows that if $\psi_1 \sim \psi_2$, then $W(\psi_1) = W(\psi_2)$. Hence we can define the weight of a pattern P to be the weight $W(\psi)$ where $\psi \in P$.

Next, since each permutation $\psi(\in G)$ on $D = \{1,2,\ldots,n\}$ can be expressed uniquely as a product of disjoint cycles, for each k = 1, 2, ..., n, we let $j_k(\psi)$ to be the number of cycles of length k in the disjoint decomposition of ψ. The cycle index of G is defined by

$$Z(G\ ;\ x_1,\ \dots,\ x_n) = \frac{1}{|G|}\sum_{\psi\in G} x_1^{j_1(\psi)}\, x_2^{j_2(\psi)} \cdots x_n^{j_n(\psi)}$$

where $x_1, \dots, x_n$ are variables.

Using Burnside's counting theorem, we can prove

Theorem 2.4 (Polya's Theorem on Pattern Counting) Let D and R be nonempty sets and let G be a permutation group on D. Suppose the weights $w(r)$ of an element r in R and $W(P)$ of a pattern P are given as above. Then the inventory of the patterns of R^D is

$$Z(G\ ;\ \sum_{r\in R} w(r),\ \sum_{r\in R}[w(r)]^2,\ \dots)$$

where $Z(G\ ;\ x_1,\ x_2,\ \dots)$ is the cycle index of G.

Corollary 2.5 If all the weights are chosen to be equal to unity, then the number of patterns of R^D is $Z(G;\ |R|,\ |R|,\ \dots,\ |R|)$.

A proof of Polya's Pattern Counting Theorem can be found in many textbooks on combinatorics, for instance, in B. Bollobás : Graph Theory-An Introductory Course, Graduate Text in Mathematics 63, Springer-Verlag, 1979; and in C. L. Liu : Introduction to Combinatorial Mathematics, McGraw-Hill, 1968.

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2. EDGE-COLOURINGS OF GRAPHS

1. Introduction and definitions

The notion of an edge-colouring of a graph can be traced back to 1880 when Tait tried to prove the Four Colour Conjecture. (A detailed account of this can be found in many existing text books on Graph Theory and therefore we shall not repeat it here.) However, there was not much development during the period 1881-1963. A breakthrough came in 1964 when Vizing proved that every graph G having maximum valency Δ can be properly edge-coloured with at most $\Delta + 1$ colours ("proper" means that no two adjacent edges of G receive the same colour). This result generalizes an earlier statement of Johnson [63] that the edges of every cubic graph can be properly coloured with four colours.

Many of the results of this chapter will be concerned with the so-called 'critical graphs' introduced by Vizing in the study of classifying which graphs G are such that $\chi'(G) = \Delta(G) + 1$. The main reference of this chapter is Fiorini and Wilson [77].

We now give a few definitions. Let G be a graph or multigraph. A *(proper, edge-) colouring* π of G is a map $\pi : E(G) \to \{1,2,...\}$ such that no two adjacent edges of G have the same image. The *chromatic index* $\chi'(G)$ of G is the minimum cardinality of all possible images of colourings of G. Hence, if $\Delta = \Delta(G)$, then it is clear that $\chi'(G) \geq \Delta$ and Vizing's theorem says that $\Delta \leq \chi'(G) \leq \Delta + 1$. If $\chi'(G) = \Delta$, G is said to be of *class 1*, otherwise G is said to be of *class 2*. If π is a colouring of G such that the image set has cardinality k, then π is called a *k-colouring* of G. If $\chi'(G) \leq k$, then G is said to be *k-colourable*. Suppose π is a k-colouring of G having image set $\{1,2,...,k\}$. Let $C_\pi(v)$ or simply $C(v)$ be the set of colours used to colour the edges incident with v and let $C'_\pi(v)$ or simply $C'(v)$ be the set $\{1,2,...,k\} - C_\pi(v)$. If $i \in C(v)$ we say that colour i is present at v. If $j \in C'(v)$, we say that colour j is absent at v.

If π is a k-colouring of G, then π decomposes E(G) into a disjoint union of colour classes $E_1,\ldots,E_k$ in such a way that for each $e \in E_i$, $\pi(e) = i$. Hence, for $i \neq j$, each connected component of $E_i \cup E_j$ is either a cycle or a chain (open chain). If $j \in C(v)$ and $i \notin C(v)$, then the connected component of $E_i \cup E_j$ containing v is said to be a $(j,i)_\pi$-chain having origin v. From the definition, each E_i is a matching in G and thus certain results on the theory of matchings can be applied to the study of edge-colourings.

We now give a brief summary of the main results of this chapter.

In §1, we prove König's theorem which says that if G is a bipartite graph or multigraph, then $\chi'(G) = \Delta(G)$. We also prove that $\chi'(K_n) = n$ if n is odd and $\chi'(K_n) = n - 1$ if n is even. These two basic results are used to determine $\chi'(O_r^t)$ for the complete t-partite graph O_r^t in §2, which in turn is used to construct a class of chromatic index critical graphs in §4.

In §2, we give a generalization of Vizing's theorem due to Andersen and Gol'dberg (Theorem 2.2). From Theorem 2.2, we deduce Vizing's theorem and some results of Ore and Shannon. We also prove several sufficient conditions for a graph G to be of class 2.

In §3, we introduce the notion of (chromatic index) critical graphs which is the main tool for classifying which graphs are of class 2. We then give several properties of critical graphs. The main result of this section is Vizing's Adjacency Lemma, which is abbreviated as VAL. The results of this section are used very often in the subsequent sections of this chapter.

In §4, we produce several methods for constructing critical graphs. The most important method is the so-called HJ-construction due to Hajós and Jakobsen. We also produce several counter-examples to the Critical Graph Conjecture (which claims that every critical graph is of odd order).

In §5, we give some lower and upper bounds on the size of critical graphs. The main tool is Fiorini's inequality (Theorem 5.3). Applying Fiorini's inequality, Vizing's conjecture on the lower bound for the size of critical graphs G is verified for $\Delta(G) \leq 4$ and is shown to be "nearly true" for $\Delta(G) = 5$ and $\Delta(G) = 6$.