1 BASIC CONCEPTS

By a *skew linear group* of degree n, a positive integer, we mean a subgroup of the general linear group GL(n,D) for some division ring D. Certain aspects of the theories of skew linear and of linear groups are very similar and in this chapter we concentrate on some of these. Other aspects are very different indeed, or at least require very different proofs.

Throughout this chapter the symbols n and D have the above designation. In the first three sections we investigate how much of the linear theory of irreducibility, absolute irreducibility, and unipotence can be extended to cover skew linear groups. Intentionally these sections are in the main more elementary than the rest of the book and we hope the reader will find them comparatively easy reading. They are also fundamental for much that follows. In the fourth and final section of Chapter 1 we construct, for later use, a wide range of examples of groups with faithful skew linear representations. This section may be omitted from a first reading.

1.1 IRREDUCIBILITY

Let G be a subgroup of GL(n,D) and set V = D^n, the space of row n-vectors over D. Then V is a D-G bimodule in the obvious way. We say that G is an *irreducible* (resp. *reducible, completely reducible*) subgroup of GL(n,D) whenever V is irreducible (resp. reducible, completely reducible) as D-G bimodule.

Viewing the elements of V as column vectors instead of row vectors, we can regard V as a G-D bimodule. It is easy to see that V is irreducible (reducible, completely reducible) as a G-D bimodule precisely when it has the property as a D-G bimodule. For let H = $\mathrm{Hom}_D(V,D)$, where

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V, D are left D-spaces. Then the right GL(n,D)-action on V and the right D-action on D make H into a GL(n,D)-D bimodule. Standard duality theory shows that V and H are isomorphic as GL(n,D)-D bimodule. If $V = V_1 \oplus V_2$ as D-G bimodule then $H \cong \mathrm{Hom}_D(V_1,D) \oplus \mathrm{Hom}_D(V_2,D)$ as G-D bimodule. Then V is D-G irreducible whenever V is G-D irreducible and the converse follows by symmetry. The claim concerning complete reducibility can be proved in a similar way. The following exercise is the basis of a less conceptual proof of these facts.

1.1.1. V *is reducible as* D-G *bimodule if and only if for some* x *in* GL(n,D) *the* (1,n)*-entry of* g^x *is* 0 *for each* $g \in G$, *if and only if* V *is reducible as* G-D *bimodule.* □

By a ***representation*** of a group G of degree n over D we mean a homomorphism ρ of G into GL(n,D). We say that ρ is irreducible (similarly with the other adjectives) whenever Gρ is irreducible, etc. We usually (but not always) put scalars on the left and mappings on the right.

Let G be a subgroup of GL(n,D) acting on $V = D^n$. Suppose there exists a series of left D-submodules of V of the form:

$$\{0\} = V_0 \leq V_1 \leq \dots \leq V_r = V \qquad (*)$$

such that G acts trivially on each factor V_i/V_{i-1} (i.e. for each i, $[V_i,G] \subseteq V_{i-1}$). Then we say that G stabilizes the series (*), and we call G a ***stability subgroup*** of GL(n,D). G is, of course, a stability group in the usual group- theoretical sense. Obvious examples are the groups $\mathrm{Tr}_1(n,D)$ (resp. $\mathrm{Tr}^1(n,D)$) of all lower (resp. upper) unitriangular matrices in GL(n,D).

Given such a series (*) for a stability group G, choose a left D-basis of V that contains a basis of each V_i. With a suitable ordering of this basis, if $x \in GL(n,D)$ is the change-of-basis matrix from the standard basis to this one, then $G^x \subseteq \mathrm{Tr}_1(n,D)$. Hence G is ***unitriangularizable*** over D. Clearly G is unipotent. The converse question of whether unipotence implies stability is discussed in Section 1.3.

Let G be a subgroup of GL(n,D) and set $V = D^n$. There exists a D-G composition series

$$\{0\} = V_0 \leq V_1 \leq \dots \leq V_r = V$$

of V. Then $W = \oplus_i V_i/V_{i-1}$ is a completely reducible D-G bimodule that is

D-isomorphic to V. The kernel of the action of G on W clearly stabilizes the composition series. We have therefore proved:

1.1.2. *Let* G *be a subgroup of* GL(n,D). *Then there exists a completely reducible representation* ρ *of* G *into* GL(n,D) *whose kernel is a stability subgroup.* □

Note that the specific construction of ρ will frequently be used below.

We now consider analogues of Maschke's, Schur's and Clifford's theorems. Care is required as certain parts of the classical arguments decidedly fail in this more general situation.

1.1.3. *Let* G *be a group and* H *a subgroup of* G *of finite index* m. *Let* D *be a division ring of characteristic* 0 *or prime to* m *and let* V *be a* D-G *bimodule which is* D-H *completely reducible. Then* V *is* D-G *completely reducible.*

Proof: Let V_1 be a D-G submodule of V. We have to show that V_1 is a D-G direct summand of V. There exists a D-H submodule U of V with $V = V_1 \oplus U$. Let μ be the natural projection of V onto U with kernel V_1 and let T be a right transversal of H to G. Set

$$\Phi = \frac{1}{m} \Sigma_{t \in T}\, t^{-1}\mu t \in \mathrm{Hom}(V,V)$$

It is easy to check that Φ is a D-G map, so $V_2 = V\Phi$ is a D-G submodule of V. Also $V_1\mu = \langle 0\rangle$ and $V_1 G \subseteq V_1$, so $V_1\Phi = \langle 0\rangle$. Further $V(\mu-1) \subseteq V_1$, so $V(\Phi-1) \subseteq V_1$. It follows that $V = V_1 \oplus V_2$. □

By setting $H = \langle 1\rangle$ in 1.1.3 we obtain the following well-known result.

1.1.4. (H. Maschke). *If* G *is a finite subgroup of* GL(n,D) *such that* charD *does not divide the order of* G, *then* G *is completely reducible.* □

So far this is exactly as for linear groups, but now something different happens: Schur's theorem (Wehrfritz [2], 1.6) does not extend.

For the construction of the following example see 1.4.24 below.

1.1.5. EXAMPLE. (Wehrfritz [13], Theorem). *Let* p *be zero or a prime. Then there exists a division ring* D, *locally finite-dimensional over its centre, and a periodic abelian subgroup* G *of* GL(2,D) *of rank* 1 *that is not completely reducible and contains no non-trivial elements of order* p. □

Obviously every subgroup of GL(1,D) = D* is irreducible. If the division ring is finite-dimensional over its centre then Schur's theorem does extend. The reader should check the proof of Schur's theorem given in Wehrfritz [2], 1.6 and spot the reason that the proof does not work more generally. The first half of Clifford's theorem, however, presents no difficulty.

1.1.6. (A. H. Clifford). *Let* G *be a group,* N *a normal subgroup of* G *and* V *an irreducible* D-G *bimodule of finite dimension over* D. *Then* V *is a direct sum of irreducible* D-N *bimodules of the same dimension over* D. *The homogeneous* D-N *components of* V *are permuted transitively by* G.

Proof: Let U be an irreducible D-N submodule of V; such exists as $\dim_D V$ is finite. Then Ug is an irreducible D-N submodule of V with $\dim_D U = \dim_D Ug$ for all $g \in G$. Also $\Sigma_{g\in G}$ Ug is a D-G submodule of V and therefore is V itself. This proves the first part.

Suppose that U_1 and U_2 are irreducible D-N submodules of V and that $\Phi : U_1 \longrightarrow U_2$ is a D-N isomorphism. Then $g^{-1}\Phi g : U_1 g \longrightarrow U_2 g$ is also a D-N isomorphism for all $g \in G$ and the second part follows. □

1.1.7. *A subnormal subgroup of a completely reducible skew linear group is completely reducible.* □

Just as in the linear case one can now define systems of imprimitivity, imprimitive groups and primitive groups and, for example, 1.9 and 1.10 of Wehrfritz [2] go through unchanged. We are unaware of any useful analogue of the main part of Clifford's theorem (Wehrfritz [2], 1.15) although much of the argument given there does have a skew linear version. We leave this to the reader.

The irreducibility or otherwise of a linear group is determined by a particular finitely generated subgroup of the group. This useful technique is not available for skew linear groups.

1.1.8. EXAMPLE. (Wehrfritz [13]). *Let* p *be a zero or a prime. Then there exists a locally finite-dimensional division ring* D *of characteristic* p *and abelian subgroups* G *and* H *of* GL(2,D) *such that* G *is not completely reducible but every finitely generated subgroup of* G *is completely reducible, and such that* H *is irreducible while every finitely generated subgroup of* H *is reducible.* □

In 1.1.8 the groups G and H can be chosen to be periodic or torsion-free. For a proof of 1.1.8 see 1.4.24 below.

If F is a field and D is a finite-dimensional F-algebra then there is no problem. For if G is a subgroup of GL(n,D) there is a finite subset X_o of G such that every element of G is an F-linear combination of elements of X_o. Set $X = \langle X_o \rangle$. Then if G is respectively irreducible, reducible or completely reducible (as a subset of GL(n,D)) then so is X, and if X is irreducible (resp. reducible, completely reducible) and if $X \leq Y \leq G$ then Y is irreducible etc, since F[G] = F[X] = F[Y]. It is tempting to try and repeat the above argument for any division ring D using linear dependence over D instead of over F, but 1.1.8 has already ruled that out. Some weakened versions, however, exist and we conclude this section with an account of them. Before that, however, we make a general but very useful remark.

1.1.9. *The matrix ring* $D^{n \times n}$ *containes at most* n *non-zero pairwise orthogonal idempotents.*

Proof: Let E be a (possibly infinite) set of non-zero pairwise orthogonal idempotents of $D^{n \times n}$. Let $V = D^n$. It is easy to see that $\Sigma_{e \in E} Ve = \oplus_{e \in E} Ve \subseteq V$, and this decomposition is as left D-module. Clearly $e \neq 0$ implies that $Ve \neq 0$. Dimension theory now implies that the cardinality of E is at most n. □

1.1.10. *Let* F *be a field and* R *an* F*-algebra that is locally semisimple Artinian (meaning that every finite subset of* R *lies in a semisimple Artinian*

F-subalgebra of R). If for some integer n there are at most n pairwise orthogonal idempotents in R, then R is semisimple Artinian.

Proof: If X is any semisimple F-subalgebra of R then X contains at most n pairwise orthogonal idempotents. Thus X_X has composition length at most n. Suppose $M_0 < M_1 < \dots < M_{n+1}$ is a chain of distinct right ideals of R. For $0 \le i \le n$ pick $x_i \in M_{i+1} \setminus M_i$. By hypothesis there exists a semisimple F-subalgebra X of R containing $\{x_0, \dots, x_n\}$. It clearly follows that $M_0 \cap X < M_1 \cap X < \dots < M_{n+1} \cap X$, which contradicts the above. Thus R is certainly Artinian. If $\underline{n}(R)$ denotes the nilradical of R then $\underline{n}(R) \cap X \subseteq \underline{n}(X) = \langle 0 \rangle$ for any semisimple subalgebra X of R, and so $\underline{n}(R) = \langle 0 \rangle$. Therefore R is semisimple. □

1.1.11. (Wehrfritz [13]). *Let G be a subgroup of* GL(n,D) *and set* $V = D^n$.

a) *If G is completely reducible then there exists a finitely generated subgroup X of G such that Y is completely reducible for every finitely generated subgroup Y of G containing X.*

b) *If G is irreducible then there is a finitely generated subgroup X of G such that for every finitely generated subgroup Y of G containing X the D-Y-bimodule V is completely reducible and homogeneous, and every irreducible D-Y submodule of V is D-X irreducible.*

The converse of Part a) is false, even in the locally finite-dimensional case; the group G of 1.1.8 shows this. Trivially the converse of Part b) is false, even if G is linear, as is shown by the group of scalar matrices for $n > 1$. The group H of 1.1.8 shows that we cannot strengthen Part b) to make X (and hence Y) irreducible, even when D is locally finite-dimensional.

Proof: Assume first that G is irreducible. Pick a finitely generated subgroup X_0 of G such that the D-X_0 composition length of $V = D^n$ is minimal. Let Y be a finitely generated subgroup of G containing X_0. Then any D-Y composition series of V, being also a series of D-X_0 submodules, is also a D-X_0 composition series for V. In particular the irreducible D-Y submodules of V are also D-X_0 irreducible. Let S_Y denote the D-Y-socle of V. The above shows that $S_Y \subseteq S_{X_0}$. Of all possible X_0

choose one X with $\dim_D S_X$ minimal. Then $S_X = S_Y$ for every finitely generated $Y \supseteq X$. Thus S_X is actually a D-G submodule of V and so $S_X = V$. Therefore for any finitely generated subgroup $Y \supseteq X$ we have $V = S_Y$; that is, Y is completely reducible. Clearly the result remains true when G is completely reducible. Part a), therefore, is established.

Finally let $X \leq Y \leq Z \leq G$ with Y, Z finitely generated. Suppose $V = V_1 \oplus \ldots \oplus V_r$, where each V_i is D-Z irreducible. By the minimality of the composition length of V as D-X bimodule, each V_i is D-X, and hence D-Y, irreducible. If H is any non-trivial homogeneous D-Y component of V then H is a sum of certain of the V_i and as such is a Z-module. This is for all such Z, so H is a non-trivial D-G submodule of V. Therefore H = V and V is D-Y homogeneous. Part b) follows. □

1.1.12. (Wehrfritz [13]). *Let* F *be a field,* D *a locally finite-dimensional division* F*-algebra and* G *a subgroup of* GL(n,D). *Set* $R = F[G] \subseteq D^{n\times n}$.

a) *If* G *is completely reducible then* R *is semisimple Artinian.*

b) *If* G *is irreducible then* R *is simple Artinian.*

If G is the group of 1.1.8, then by 1.1.12 a) and 1.1.10 the ring R is semisimple Artinian. Thus the converse of 1.1.12 a) is false. In fact the ring R is simple. For if e is a central idempotent of R then $V = D^2 = Ve \oplus V(1-e)$, and so if R is not simple then G is completely reducible, which is not true. Hence R is simple, which shows that the converse of 1.1.12 b) is also false. A less interesting counter-example to the converse of 1.1.12 b) is furnished by the group $G = \langle \operatorname{diag}(\alpha,1) : \alpha \in D^* \rangle$ for any division ring D. For $R \cong D$ is simple and yet G is reducible.

Note also that 1.1.12 is not true for division rings in general. For suppose that D is a division ring with a central subfield F and that $x \in D$ is not algebraic over F. Set $G = \langle x \rangle \leq D^* = GL(1,D)$. Then G is clearly irreducible and yet $F[G] \cong F[x,x^{-1}]$ is not Artinian.

Proof: a) First assume that G is finitely generated and completely reducible. Then R has finite dimension over F. Let W be an irreducible D-G submodule of $V = D^n$ and pick $w \in W\backslash\langle 0\rangle$. Then wR contains an irreducible R-module W_1, say.

Clearly $W = DW_1$ and $dW_1 \cong_R W_1$ for all $d \in D^*$. Thus W is completely reducible as R-module. But V is D-G completely reducible, so V is completely reducible as R-module. Thus so too is $D^{n\times n}$ as a right R-module, and it follows that R is semisimple Artinian.

In the general case choose the finitely generated subgroup X of G as in 1.1.11 a). Then by what we have already proved the set $\{ F[Y] : X \leq Y \leq G$ with Y finitely generated $\}$ is a local system of semi-simple Artinian F-subalgebras of R. By 1.1.9 the ring R contains at most n orthogonal idempotents, so R is semisimple Artinian by 1.1.10. Part a) is, therefore, established.

b) By Part a) R is semisimple Artinian, so $V = D^n$ contains an irreducible R-module W, say. Clearly $V = DW$ and $dW \cong_R W$ for every $d \in D^*$. Thus V is homogeneous and faithful as R-module. It follows that R is simple. □

In the opposite direction we have.

1.1.13. (Wehrfritz [13]). *Let* F *be a field,* D *a division* F*-algebra and* G *a subgroup of* GL(n,D). *Set* $R = F[G] \subseteq D^{n\times n}$.

a) *If* R *is semiprime (e.g. if* R *is semisimple Artinian) then* G *is isomorphic to a completely reducible subgroup of* GL(n,D).

b) *If* R *is simple Artinian then for some* $m \leq n$ *the group* G *is isomorphic to an irreducible subgroup of* GL(m,D).

Proof: a) Let $\langle 0\rangle = V_0 < V_1 < \ldots < V_r = V$ be a D-G composition series of $V = D^n$. Set $H = \bigcap_i C_G(V_i/V_{i-1})$. As in 1.1.2 the group G/H is isomorphic to a completely reducible subgroup of GL(n,D). Also $H \subseteq 1 + \underline{n}$ where $\underline{n} = \{ x \in R : V_i x \subseteq V_{i-1}$ for each i $\}$. But $\underline{n}^r = \langle 0\rangle$ and so $\underline{n}$ is a nilpotent ideal of the semiprime ring R. Thus $\underline{n} = \langle 0\rangle$ and Part a) follows.

b) Let W be an irreducible D-G submodule of V. Since R is Artinian W contains an irreducible R submodule W_1, say. The simplicity of R implies that W_1, and thus also W, is a faithful R-module. The result follows with $m = \dim_D W$. □

As a weak generalization of 1.1.12 and a converse of 1.1.13 we have the following .

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1.1.14. *Let F, D, G and R be as in 1.1.13.*

a) *If G is completely reducible then R is semiprime; more generally so is F[N] for every subnormal subgroup N of G.*

b) *If G is irreducible then R is prime.*

c) *If G is locally finite and completely reducible then R is semisimple Artinian.*

Proof: Let $V = D^n$. Since F is central the D-G and D-R submodules of V coincide.

a) Let $\underline{a}$ be an ideal of R with $\underline{a}^2 = \langle 0 \rangle$. By hypothesis $V = V\underline{a} \oplus W$ for some D-G submodule W of V. But then

$$V\underline{a} = V\underline{a}^2 \oplus W\underline{a} = W\underline{a} \subseteq V\underline{a} \cap W = \langle 0 \rangle$$

Consequently $\underline{a} = \langle 0 \rangle$ and R is semiprime. By Clifford's Theorem (1.1.7), N is also completely reducible, so F[N] is also semiprime.

b) Suppose $\underline{a}$ and $\underline{b}$ are ideals of R with $\underline{ab} = 0$. Then $V\underline{a}$ is a D-G submodule of V, and so is either V or $\langle 0 \rangle$. If $V\underline{a} = V$ then $V\underline{b} \subseteq V\underline{ab} = \{0\}$ and so $\underline{b} = \{0\}$. If $V\underline{a} = \{0\}$ then $\underline{a} = \{0\}$. Therefore R is prime.

c) By 1.1.11 there is a finite subgroup X of G such that every finite subgroup Y of G containing X is completely reducible. Then the subalgebra F[Y] of R is semiprime by Part a) and finite-dimensional. Thus each F[Y] is semisimple Artinian, and consequently so is R, by 1.1.9 and 1.1.10. □

Occasionally we need to deal with subrings of $D^{n \times n}$ which are not of the form F[G] for some $G \leq GL(n,D)$. They will, however, be normalized by G. The most common type to arise has the form K[N] where K is some non-central subfield of $D^{n \times n}$ normalized by G and N is a normal subgroup of G. Then following result will then be helpful.

1.1.15. *Let G be an irreducible subgroup of* GL(n,D) *and S a subring of* $D^{n \times n}$ *normalized by G. Then S is semiprime.*

S need not be prime, for let G be the full monomial group for $n > 1$ and S the subring of all diagonal matrices. Also if G is only completely reducible then S need not be semiprime, for example if $G = \langle 1 \rangle$ and S is the subring of lower triangular matrices for $n > 1$. (In the latter

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case S is semiprime if S = F[N] for some N normal in G, by 1.1.14.)

Proof: Let $\underline{n}$ be the upper nilradical of S and $V = D^n$. Then $\underline{n}$ is nilpotent (Chatters & Hajarnavis [1], 1.35, or see 1.3.9) and normalized by G. Suppose $\underline{n}^r \neq \underline{n}^{r+1} = \{0\}$. Then $V\underline{n}^r$ is a non-zero D-G submodule of V and hence is V. But then $\underline{n}$ annihilates V and so $\underline{n} = \{0\}$, as required. □

1.2 ABSOLUTE IRREDUCIBILITY

Suppose G is a subgroup of GL(n,F) for some field F. Then G is absolutely irreducible if one of the following hold: G is irreducible over the algebraic closure of F ; $F[G] = F^{n\times n}$; G is irreducible over every extension field of F; $C_{F^{n\times n}}(G) = F1_n$ (Wehrfritz [2], 1.18). The concept of absolute irreducibility for skew linear groups is much more important than that of irreducibility but it is perhaps not obvious what the correct definition should be.

Let D be a division ring with centre F. A subgroup G of GL(n,D) is ***absolutely irreducible over*** D if $F[G] = D^{n\times n}$. This clearly extends the definition for linear groups. Also the following is obvious.

1.2.1. *If* $G \leq GL(n,D)$ *is absolutely irreducible over* D *then* G *is irreducible over* D. □

Suppose that G is an absolutely irreducible subgroup of GL(n,D). Let F' be any extension field of F. Then $R' = F' \otimes_F D^{n\times n}$ is a simple F'-algebra (Cohn [2], p. 364). If $\dim_F D$ is finite then clearly $\dim_{F'} R'$ is finite and R' is Artinian. If $\dim_F F'$ is finite then R' is finitely generated as right $D^{n\times n}$-module and again R' is Artinian. If R' is Artinian then R' is a matrix ring of degree n', say, over some division F'-algebra D' by the Artin-Wedderburn theorem. Also clearly $G \cong 1 \otimes G \leq R'$ and $F'[G] = R'$. Hence we have the following result.

1.2.2. *Let* G *be an absolutely irreducible subgroup of* GL(n,D) *and let* F' *be an extension field of* F *with either* $\dim_F D$ *or* $\dim_F F'$ *finite. Then there is a natural way of regarding* G *as an absolutely irreducible subgroup of* $GL(n',D')$ *for some integer* n' *and division* F'*-algebra* D'. □

