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Triangulated Categories in the Representation Theory of Finite Dimensional Algebras

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PREFACE

The aim of these notes is to show that the concept of triangulated categories might be useful for studying modules over finite-dimensional k-algebras, where k is a field which for technical reasons usually will be assumed to be algebraically closed. On the other hand we try to show that certain triangulated categories become quite accessible if one applies methods of representation theory emerged in recent years.

Let $\mathcal{A}$ be an abelian category, then the most famous example of a triangulated category is the derived category $D^b(\mathcal{A})$ of bounded complexes over $\mathcal{A}$. Usually we will be interested in the case that $\mathcal{A}$ is the category mod A of finitely generated left modules over a finite-dimensional k-algebra. In this case the category $D^b(\mathcal{A}) = D^b(\text{mod } A)$ may be identified with the homotopy category $K^{-,b}({}_A\mathcal{P})$ of complexes bounded above over the finitely generated projective left A-modules having bounded cohomology. Since their introduction in the sixties they have turned out quite useful in algebraic geometry and homological algebra. Examples for this can be found in duality theory (Hartshorne (1966), Iversen (1986)) or in the fundamental work on perverse sheaves by Bernstein, Beilinson, Deligne (1986).

The concept of derived categories goes back to suggestions of Grothendieck. We refer to Grothendieck (1986) for some information about the motivations and developments. The formulation in terms of triangulated categories was achieved by Verdier in the sixties and an account of this is published in Verdier (1977). We will assume that the reader is familiar with the concept of localization which is needed in the construction of the derived category. Information about this may be found for example in Verdier (1977), Hartshorne (1966) or Iversen (1986).

In recent years there has been quite some interest in the structure of the category mod A of finitely generated left modules over a finite-dimensional k-algebra A. (Unless stated otherwise the term module always refers to a finitely generated left A-module). We will assume some elementary facts from the general theory which are easily accessible in textbooks such as Anderson, Fuller (1974) or Curtis, Reiner (1962), (1981). For more andvanced results we refer to Ringel (1984). Also we adopt the categorical language of Mac Lane (1971) and use rather standard results from homological algebra for which Cartan, Eilenberg (1956) serves as a reference.

There are two classes of finite-dimensional k-algebras which have been studied quite intensively. These are the finite-dimensional hereditary k-algebras (i.e. submodules of projective modules are projective) and the selfinjective finite-dimensional k-algebras (i.e. the algebra considered as a module is an injective module). It seems that there is a strong relationship between these two classes of algebras. Some evidence for this is contained in chapter V.

We now turn to a description of the content of the various chapters.

Chapter I serves as an introduction to the theory of triangulated categories. We introduce Frobenius categories and show that the associated stable categories admit a triangulated structure. It is customary in representation theory to associate a quiver (directed graph) $\vec{\Gamma}(\mathcal{a})$ to an additive category $\mathcal{a}$. In a lot of cases this quiver has the additional structure of a translation quiver. A celebrated example is the case of $\mathcal{a} = \mathrm{mod}\, A$. Motivated by the work of Auslander and Reiten we introduce Auslander-Reiten triangles in a triangulated category and show that $D^b(A)$ has Auslander-Reiten triangles in case A has finite global dimension. This result endows $\vec{\Gamma}(D^b(A))$ with the structure of a translation quiver. If A is a hereditary finite dimensional k-algebra this quiver is calculated explicitly.

In chapter II we show that the derived category $D^b(A)$ for a finite-dimensional k-algebra of finite global dimension can be interpreted in quite some different way. There is a classical construction to associate with a given finite-dimensional k-algebra A a finite-dimensional selfinjective k-algebra. The injective cogenerator $DA = \mathrm{Hom}_k(A,k)$ admits a natural A-bimodule structure. So we may form the trivial extension algebra $T(A)$ of A by DA. The algebra $T(A)$ is a $\mathbb{Z}$-graded algebra and the category $\mathrm{mod}^{\mathbb{Z}}T(A)$ of finitely generated $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree zero is a Frobenius category. (For different descriptions of $\mathrm{mod}^{\mathbb{Z}}T(A)$ we refer to chapter II.2). In particular we obtain that the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}T(A)$ is a triangulated category. The main theorem now asserts that there exists a full and faithful functor $F : D^b(A) \to \underline{\mathrm{mod}}^{\mathbb{Z}}T(A)$ preserving the triangulated structure which is dense if A has finite global dimension.

The third chapter gives a self-contained approach to tilting theory. Some of the history is recalled in remarks preceding this chapter. It is independent of chapter II and most of chapter I. We have included most of the theoretical results available. In section 5 some of the results are stated without proof in order to avoid some overlap with Ringel (1984). For related results, not treated explicitly, we have included references and some comments. We regret any incompleteness in those.

In chapter IV we concentrate on a rather special class of finite-dimensional k-algebras. We assume that their derived categories have a predescribed form, namely they are equivalent to the derived category of a finite-dimensional hereditary k-algebra. This has a lot of consequences. For example a representation-finite (there are up to isomorphism only finitely many indecomposable modules) algebra in this class has the property that the indecomposable modules are uniquely (up to isomorphism) determined by their composition factors. The main result of this chapter asserts that such an algebra is already tiltable (see IV.4 for a definition) to a hereditary finite-dimensional k-algebra.

The final chapter relates the results from II.4 and IV.5 to give a description of the representation-finite trivial extension algebras.

With a few exceptions (for example II.5, III.5, IV.6, V.2) the needed results are contained in the references given above. These exceptions are usually not essential in the subsequent developments. We hope that this and the additional references will stimulate the reader for further studies in this area.

It is my pleasure to thank I. Assem, S. Brenner, C.M. Ringel and L. Unger for animating discussions on various aspects of the theory presented here. My special thanks go to Mrs Köllner for her splendid job in typing the manuscript. Also I thank Cambridge University Press for publishing it in this series and for checking the English.