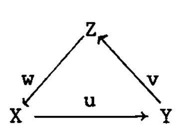
CHAPTER I TRIANGULATED CATEGORIES

1. Foundations

The basic reference for triangulated categories and derived categories is the original article of Verdier (1977). Also Hartshorne (1966), Beilinson, Bernstein and Deligne (1982) and Iversen (1986) give introductions to these concepts.

Let us remind the reader of our convention that the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathcal{K}$ is denoted by fg. Unless otherwise stated, we adopt the categorical language of Mac Lane (1971). In particular, our additive categories have finite direct sums.

1.1 Let $\mathcal{C}$ be an additive category and T an automorphism of $\mathcal{C}$. The automorphism T is usually called the <u>translation functor</u>. The inverse of T is denoted by T^{-}. A <u>sextuple</u> (X,Y,Z,u,v,w) in $\mathcal{C}$ is given by objects $X,Y,Z \in \mathcal{C}$ and morphisms $u : X \to Y$, $v : Y \to Z$ and $w : Z \to TX$. A more suggestive notation of sextuples is:
$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$.
(The following notation for a sextuple is also used sometimes in the literature:



But keep in mind that w is a morphism from Z to TX. We will not use this notation, except in the situation when we explain the terminology of the axiom (TR4) of a triangulated category).

A <u>morphism of sextuples</u> from (X,Y,Z,u,v,w) to (X',Y',Z',u',v',w') is a triple (f,g,h) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

If in this situation f,g and h are isomorphisms in $\mathcal{C}$, the morphism is then called an isomorphism.

A set $\mathcal{T}$ of sextuples in $\mathcal{C}$ is called a <u>triangulation</u> of $\mathcal{C}$ if the following conditions are satisfied. The elements of $\mathcal{T}$ are then called <u>triangles</u>.

(TR1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $u : X \to Y$ in $\mathcal{C}$ can be embedded into a triangle (X,Y,Z,u,v,w). The sextuple $(X,X,0,1_X,o,o)$ is a triangle. (1_X denotes the identity morphism from X to X).

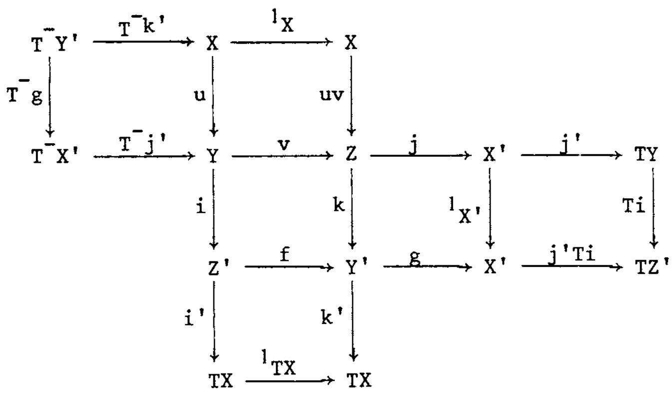
(TR2) If (X,Y,Z,u,v,w) is a triangle, then $(Y,Z,TX,v,w,-Tu)$ is a triangle.

(TR3) Given two triangles (X,Y,Z,u,v,w) and (X',Y',Z',u',v',W') and morphisms $f : X \to X'$, $g : Y \to Y'$ such that $fu' = ug$, there exists a morphism (f,g,h) from the first triangle to the second.

(TR4) (The octahedral axiom)

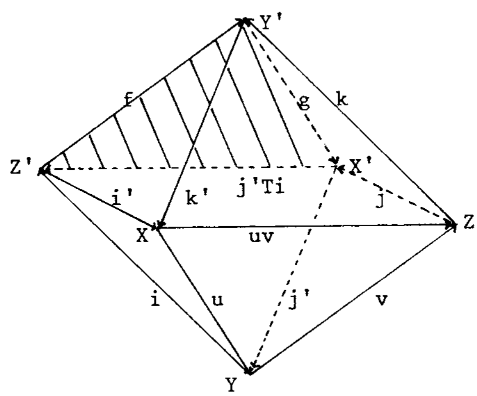
Consider triangles (X,Y,Z',u,i,i'), (Y,Z,X',v,j,j') and

$(X,Z,Y',u \cdot v,k,k')$. Then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle.



The additive category $\mathcal{C}$ together with a translation functor T and a triangulation $\mathcal{T}$ is called a <u>triangulated</u> <u>category</u>.

A different way of displaying the axiom (TR4) is the following: Given triangles as above. Then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the hatched face of the following octahedron is a triangle and $if = vk$, $fk' = i'$, $kg = j$ and $k'Tu = gj'$. (Observe that the last conditions can be expressed by the requirement that $(1_X,v,f)$ is a morphism from the first triangle to the third and that $(u,1_Z,g)$ is a morphism from the third triangle to the second).



Let $\mathcal{C} = (C,T,\mathcal{T})$, $\mathcal{C}' = (C,T',\mathcal{T}')$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called exact if there exists an invertible natural transformation $\alpha : FT \to T'F$ such that $(FX,FY,FZ,Fu,Fv,Fw\alpha_X)$ is in $\mathcal{T}'$ whenever (X,Y,Z,u,v,w) is in $\mathcal{T}$.

If an exact functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories, we call it a triangle-equivalence. $\mathcal{C}$ and $\mathcal{C}'$ are then called triangle-equivalent. The following lemma is straightforward.

LEMMA. Let $F : \mathcal{C} \to \mathcal{C}'$ be a triangle-equivalence and let G be a quasi-inverse of F. Then G is a triangle-equivalence.

An additive functor $H : \mathcal{C} \longrightarrow \mathcal{A}$ from a triangulated category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is called a (covariant) cohomological functor, if whenever (X,Y,Z,u,v,w) is a triangle, the long sequence

$$\cdots \to H(T^iX) \xrightarrow{H(T^iu)} H(T^iY) \xrightarrow{H(T^iv)} H(T^iZ) \xrightarrow{H(T^iw)} H(T^{i+1}X) \to \cdots$$

is exact. The additive functor $H : \mathcal{C} \to \mathcal{A}$ is called a (contravariant) cohomological functor, if whenever (X,Y,Z,u,v,w) is a triangle, the long sequence

$$\cdots \to H(T^{i+1}X) \xrightarrow{H(T^iu)} H(T^iZ) \xrightarrow{H(T^iv)} H(T^iY) \xrightarrow{H(T^iw)} H(T^i(X)) \to \cdots$$

is exact.

1.2 In the following proposition we collect some elementary properties.

PROPOSITION. Let $\mathcal{C}$ be a triangulated category, (X,Y,Z,u,v,w) be a triangle and M be an object in $\mathcal{C}$. Then

(a) $uv = vw = 0$.

(b) $\mathrm{Hom}_{C}(M,-)$ and $\mathrm{Hom}_{C}(-,M)$ are cohomological functors.

(c) Let (f,g,h) be a morphism of triangles from (X,Y,Z,u,v,w) to (X',Y',Z',u',v',w'). If f and g are isomorphisms, then so is h.

<u>Proof:</u> (a) In virtue of (TR2) it is enough to show that $uv = o$. By (TR2) we know that $(Y,Z,TX,v,w,-Tu)$ is a triangle. By (TR3) the following diagram can be completed to a commutative diagram:

$$\begin{array}{ccccccc}
Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle 1_Z} & & \downarrow & & \downarrow{\scriptstyle Tv} \\
Z & \xrightarrow{1_Z} & Z & \xrightarrow{0} & 0 & \xrightarrow{0} & TZ
\end{array}$$

In particular $(-Tu)Tv = 0$, hence $uv = 0$.

(b) We show that $\mathrm{Hom}_{C}(M,-)$ is exact. The other assertion follows dually. It is enough to show that

$$\mathrm{Hom}(M,T^{i}X) \xrightarrow{\mathrm{Hom}(M,T^{i}u)} \mathrm{Hom}(M,T^{i}Y) \xrightarrow{\mathrm{Hom}(M,T^{i}v)} \mathrm{Hom}(M,T^{i}Z)$$

is exact. By (a) we know that $\mathrm{Hom}(M,T^{i}u)\cdot\mathrm{Hom}(M,T^{i}v) = 0$. So let $g \in \mathrm{Hom}(N,T^{i}Y)$ such that $gT^{i}v = 0$. By (TR1) and (TR2) we obtain that the rows of the following diagram are triangles

$$\begin{array}{ccccccc}
T^{-}M & \xrightarrow{o} & 0 & \xrightarrow{o} & T^{-i+1}M & \xrightarrow{-1_{T^{-i+1}M}} & T^{-i+1}M \\
{\scriptstyle T^{-i}g}\downarrow & & \downarrow{\scriptstyle o} & & & & \downarrow{\scriptstyle T^{-i+1}g} \\
Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY
\end{array}$$

By assumption $T^{-i}gv = 0$. By (TR3) there exists h such that $hTu = T^{-i+1}g$. Hence $(T^{i-1}h)T^{i}u = g$.

(c) Consider the following commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

and assume that f and g are isomorphisms.

We apply the cohomological functor $\mathrm{Hom}(Z',-)$ and obtain the following commutative diagram which has exact rows by (b).

$$\begin{array}{ccccccccc} \mathrm{Hom}(Z',X) & \to & \mathrm{Hom}(Z',Y) & \to & \mathrm{Hom}(Z',Z) & \to & \mathrm{Hom}(Z',TX) & \to & \mathrm{Hom}(Z',TY) \\ \downarrow \mathrm{Hom}(Z',f) & & \downarrow \mathrm{Hom}(Z',g) & & \downarrow \mathrm{Hom}(Z',h) & & \downarrow \mathrm{Hom}(Z',Tf) & & \downarrow \mathrm{Hom}(Z',Tg) \\ \mathrm{Hom}(Z',X') & \to & \mathrm{Hom}(Z',Y') & \to & \mathrm{Hom}(Z',Z') & \to & \mathrm{Hom}(Z',TX') & \to & \mathrm{Hom}(Z',TY') \end{array}$$

Since $\mathrm{Hom}(Z',f)$, $\mathrm{Hom}(Z',g)$, $\mathrm{Hom}(Z',Tf)$ and $\mathrm{Hom}(Z',Tg)$ are isomorphisms, we infer that $\mathrm{Hom}(Z',h)$ is an isomorphism.

In particular there exists $\varphi \in \mathrm{Hom}(Z',Z)$ such that $\varphi h = 1_{Z'}$. Conversely, apply the cohomological functor $\mathrm{Hom}(-,Z)$. As above we conclude that $\mathrm{Hom}(h,Z)$ is an isomorphism. In particular there exists $\psi \in \mathrm{Hom}(Z',Z)$ such that $h\psi = 1_Z$. Hence h is an isomorphism.

1.3 The following lemma shows that the converse of (TR2) is satisfied. This shows that we are dealing with the classical definition of a triangulated category.

LEMMA. If $(Y,Z,TX,v,w,-Tu)$ is a triangle, then (X,Y,Z,u,v,w) is a triangle.

Proof: If $(Y,Z,TX,v,w,-Tu)$ is a triangle we obtain by applying (TR2) that $(TX,TY,TZ,-Tu,-Tv,-Tw)$ is a triangle. By (TR1) there exists a triangle (X,Y,Z',u,v',w'), hence $(TX,TY,TZ',-Tu,-Tv',-Tw')$ is a triangle by (TR2), Now consider the following diagram:

$$\begin{array}{ccccccc}
TX & \xrightarrow{-Tu} & TY & \xrightarrow{-Tv} & TZ & \xrightarrow{-Tw} & T^2X \\
\| & & \| & & \downarrow h & & \| \\
TX & \xrightarrow{-Tu} & TY & \xrightarrow{-Tv'} & TZ' & \xrightarrow{-Tw'} & T^2X
\end{array}$$

The morphism h exists by (TR3) which moreover is an isomorphism by 1.2 c). So we obtain the following commutative diagram, where the second row is a triangle and the vertical morphisms are isomorphisms.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\| & & \| & & \downarrow T^{-}h & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX
\end{array}$$

By (TR1) we infer that the first row is a triangle.

1.4 LEMMA. Let $\mathcal{C}$ be a triangulated category and (X,Y,Z,u,v,w) be a triangle. The following are equivalent:

(i) $w = o$.

(ii) u is a section.

(iii) v is a retraction.

Proof: If $w = o$ consider the following morphism of triangles. The existence of u' is guaranteed by (TR3).

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\| & & \downarrow u' & & \downarrow o & & \| \\
X & \xrightarrow{1_X} & X & \xrightarrow{o} & 0 & \xrightarrow{o} & TX
\end{array}$$

Thus $uu' = 1_X$, hence u is a section.

Conversely, if u is a section, there exists u' such that $(1_X,u',o)$

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is a morphism from the triangle (X,Y,Z,u,v,w) to the triangle $(X,X,0,1_X,o,o)$. In particular $w 1_{TX} = o$, hence $w = o$. In the same way one can show that (i) and (iii) are equivalent.

1.5 LEMMA. Let C be a triangulated category and (X,Y,Z,u,v,w) be a triangle. The following are equivalent:

(i) If $f : W \to Z$ is not a retraction, then there exists $f' : W \to Y$ such that $f'v = f$.

(ii) If $f : W \to Z$ is not a retraction, then $fw = o$.

Proof: By 1.2 (b) we know that

$$\mathrm{Hom}(W,Y) \xrightarrow{\mathrm{Hom}(W,v)} \mathrm{Hom}(W,Z) \xrightarrow{\mathrm{Hom}(W,w)} \mathrm{Hom}(W,TX)$$ is exact.

1.6 LEMMA. Let C be a triangulated category and (X,Y,Z,u,v,w) and (X',Y',Z',u',v',w') be triangles in C. Let $g : Y \to Y'$ be a morphism. Then the following are equivalent:

(i) $ugv' = o$.

(ii) There exists a morphism (f,g,h) from the first triangle to the second.

If these conditions are satisfied and $\mathrm{Hom}(X,T^{-}Z') = 0$ then f and h are uniquely determined by g.

Proof: We apply $\mathrm{Hom}_C(X,-)$ to the second triangle and obtain an exact sequence $\mathrm{Hom}(X,T^{-}Z') \to \mathrm{Hom}(X,X') \to \mathrm{Hom}(X,Y') \to \mathrm{Hom}(X,Z')$. If $ugv' = o$, there exists $f : X \to X'$ such that $fu' = ug$. By (TR3) we obtain (i) $\Rightarrow$ (ii).

Conversely, if (f,g,h) is a morphism then $ugv' = fu'v' = 0$ by 1.2 a. If $\mathrm{Hom}(X,T^{-}Z') = 0$, then $\mathrm{Hom}(X,u')$ is a monomorphism, which shows that f is uniquely determined by g. If $\mathrm{Hom}(X,T^{-}Z') = 0$, then also $\mathrm{Hom}(TX,Z') = 0$. Thus $\mathrm{Hom}(v,Z')$ is a monomorphism, which shows that h is uniquely determined by g.

1.7 LEMMA. Let $\mathcal{C}$ be a triangulated category. Then $(X,Y,0,u,o,o)$ is a triangle if and only if u is an isomorphism.

Proof: If u is an isomorphism, then $(X,Y,0,u,o,o)$ is isomorphic to $(Y,Y,0,1_Y,o,o)$. Thus the assertion follows from (TR1). If $(X,Y,0,u,o,o)$ is a triangle, we consider the following morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{o} & 0 & \xrightarrow{o} & TX \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1_Y} & & \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle Tu} \\
Y & \xrightarrow{1_Y} & Y & \longrightarrow & 0 & \longrightarrow & TY
\end{array}$$

Thus the assertion follows from 1.2.

2. Frobenius categories

2.1 Let $\mathcal{B}$ be an additive category embedded as a full and extension-closed subcategory in some abelian category $\mathcal{A}$. Let $\mathcal{S}$ be the set of exact sequences in $\mathcal{A}$ with terms in $\mathcal{B}$. Following Quillen (1973) the pair $(\mathcal{B},\mathcal{S})$ is called an <u>exact category</u>. A morphism $u : X \to Y$ in $\mathcal{B}$ is called a <u>proper monomorphism</u> if there exists an exact sequence $0 \to X \overset{u}{\to} Y \to Z \to 0$ in $\mathcal{S}$. A morphism $v : Y \to Z$ in $\mathcal{B}$ is called a <u>proper epimorphism</u> if there exists an exact sequence $0 \to X \to Y \overset{v}{\to} Z \to 0$ in $\mathcal{S}$.

An object P in $\mathcal{B}$ is called $\mathcal{S}$-<u>projective</u> if for all proper epimorphisms $v : Y \to Z$ and morphisms $f : P \to Z$ in $\mathcal{B}$ there exists $g : P \to Y$ such that $f = gv$.

An object I in $\mathcal{B}$ is called $\mathcal{S}$-<u>injective</u> if for all proper monomorphisms $u : X \to Y$ and morphisms $f : X \to I$ in $\mathcal{B}$ there exists $g : Y \to I$ such that $f = ug$.

The following two assertions are straightforward.

<u>LEMMA</u>. Let $(\mathcal{B},\mathcal{S})$ be an exact category. Then the following are equivalent:

(i) $P \in \mathcal{B}$ is $\mathcal{S}$-projective.

(ii) All exact sequences $0 \to X \to Y \to P \to 0$ in $\mathcal{S}$ split.