1

GROUP REPRESENTATIONS

1.1. Introduction

One of the origins of group theory stems from the observation that certain operations, such as permutations, linear transformations and maps of a space onto itself, permit of a law of composition that is analogous to multiplication. Thus the early work was concerned with what we may call concrete groups, in which the 'product' of two operations can be computed in every instance.

It was much later that group theory was developed from an axiomatic point of view, when it was realised that the structure of a group does not significantly depend upon the nature of its elements.

However, it is sometimes profitable to reverse the process of abstraction. This is done by considering homomorphisms

$$\theta\colon G \to \Gamma,$$

where G is an abstract group and Γ is one of the concrete groups mentioned above. Such a homomorphism is called a *representation of G*. Accordingly, we speak of representations by permutations, matrices, linear transformations and so on.

One of the oldest examples of a permutation representation is furnished by Cayley's Theorem, which states that a finite group

$$G: x_1(=1), x_2, \ldots, x_g$$

can be represented as a group of permutations of degree g, that is by permutations acting on g objects. In this case, the objects are the elements of G themselves. With a typical element x of G we associate the permutation

$$\pi(x) = \begin{pmatrix} x_1 & x_2 & \ldots & x_g \\ x_1x & x_2x & \ldots & x_gx \end{pmatrix}; \tag{1.1}$$

this is indeed a permutation, because the second row in (1.1) consists of all the elements of G in some order. More briefly, we shall write

$$\pi(x)\colon x_i \to x_ix \quad (i = 1, 2, \ldots, g).$$

If y is another element of G, we have analogously

$$\pi(y)\colon x_i \to x_i y \quad (i = 1, 2, \ldots, g).$$

In this book the product of permutations is interpreted as a sequence of instructions read from left to right. Thus $\pi(x)\pi(y)$ signifies the operation whereby a typical element x_i of G is first multiplied by x and then by y on the right, that is

$$\pi(x)\pi(y)\colon x_i \to (x_i x)y \quad (i = 1, 2, \ldots, g).$$

Since this is the same operation as

$$\pi(xy)\colon x_i \to x_i(xy) \quad (i = 1, 2, \ldots, g),$$

we have established the crucial relationship

$$\pi(x)\pi(y) = \pi(xy), \tag{1.2}$$

which means that the map

$$\pi\colon G \to S_g$$

is a homomorphism of G into the symmetric group S_g, the group of all permutations on g symbols. This homomorphism, which is the content of Cayley's Theorem, is called the *right-regular representation* of G.

A given group G may have more than one representation by permutations, possibly of different degrees. Suppose that H is a subgroup of G of finite index n, and let

$$G = Ht_1 \cup Ht_2 \cup \ldots \cup Ht_n$$

be the coset decomposition of G relative to H. With a typical element x of G we associate the permutation

$$\sigma(x) = \begin{pmatrix} Ht_1 & Ht_2 & \ldots & Ht_n \\ Ht_1 x & Ht_2 x & \ldots & Ht_n x \end{pmatrix}, \tag{1.3}$$

in which the permuted objects are the n cosets. As before, it can be verified that

$$\sigma(x)\sigma(y) = \sigma(xy),$$

which proves that the map

$$\sigma\colon G \to S_n$$

is a homomorphism of G into S_n.

These examples serve to illustrate the notion of a permutation representation. For the remainder of the book we shall be concerned almost exclusively with homomorphisms

$$A: G \to GL_m(K), \tag{1.4}$$

where $GL_m(K)$ is the general linear group of degree m over K, that is the set of all non-singular $m \times m$ matrices with coefficients in a given *ground field* K. The integer m is called the *degree* (or *dimension*) of the representation A. We describe the situation formally as follows:

Definition 1.1. *Suppose that with each element x of the group G there is associated an m by m non-singular matrix*

$$A(x) = (a_{ij}(x)) \quad (i, j = 1, 2, \ldots, m),$$

with coefficients in the field K, in such a way that

$$A(x)A(y) = A(xy) \quad (x, y \in G). \tag{1.5}$$

Then $A(x)$ is called a matrix representation of G of degree (dimension) m over K.

A brief remark about nomenclature is called for: in Analysis we frequently speak of a 'function $f(x)$', when we should say 'a function (or map) f which assigns the value $f(x)$ to x'. We are here indulging in a similar abuse of language and refer to 'the representation $A(x)$' instead of using the more correct but clumsy phrase 'the homomorphism $A: G \to GL_m(K)$ which assigns to x the matrix $A(x)$'. When it is convenient, we abbreviate this to 'the representation A'.

Some consequences of (1.5) may be noted immediately. Let $x = y = 1$. Then we have that

$$\{A(1)\}^2 = A(1).$$

Since $A(1)$ is non-singular, it follows that

$$A(1) = I,$$

the unit matrix of dimension m. Next, put $y = x^{-1}$. Then

$$A(x)A(x^{-1}) = I,$$

so that

$$A(x^{-1}) = (A(x))^{-1}. \tag{1.6}$$

We emphasise that a representation A need not be injective ('one-to-one'), that is it may happen that $A(x) = A(y)$ while $x \neq y$. The kernel of A consists of those elements u of G for which $A(u) = I$. The kernel is always a normal, possibly the trivial, subgroup of G [**13**, p. 67]. The representation is injective or *faithful* if and only if the kernel reduces to the trivial group $\{1\}$. For the equation $A(x) = A(y)$ is equivalent to

$$A(x)(A(y))^{-1} = A(xy^{-1}) = I,$$

and for a faithful representation this implies that $xy^{-1} = 1$, that is $x = y$.

When $m = 1$, the representation is said to be *linear*. In this case we identify the matrix with its sole coefficient. Thus a linear representation is a function on G with values in K, say

$$\lambda: G \to K$$

such that

$$\lambda(x)\lambda(y) = \lambda(xy). \tag{1.7}$$

Every group possesses the *trivial* (formerly called *principal*) *representation* given by the constant function

$$\lambda(x) = 1 \quad (x \in G). \tag{1.8}$$

A non-trivial example of a linear representation is furnished by the *alternating character* of the symmetric group S_n (for each $n > 1$). This is defined by

$$\zeta(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

The eqaation

$$\zeta(x)\zeta(y) = \zeta(xy)$$

expresses a well-known fact about the parity of permutations [**13**, p. 134].

Let $A(x)$ be a representation of G and suppose that

$$B(x) = T^{-1}A(x)T, \tag{1.9}$$

where T is a fixed non-singular matrix with coefficients in K. It is readily verified that

$$B(x)B(y) = B(xy),$$

so that $B(x)$, too, is a representation of G. We say the representations $A(x)$ and $B(x)$ are *equivalent over* K, and we write

$$A(x) \sim B(x).$$

In the relationship (1.9) the exact form of T is usually irrelevant, but it is essential that its coefficients lie in K. As a rule, we do not distinguish between equivalent representations, that is we are only interested in equivalence classes of representations.

1.2. *G*-modules

The notion of equivalence becomes clearer if we adopt a more geometric approach. We recall the concept of a linear map

$$\alpha: V \to W$$

between two vector spaces over K. Under this map the image of a vector $\mathbf{v}$ of V will be denoted by $\mathbf{v}\alpha$, the operator α being written on the right. The map is *linear* if for all $\mathbf{u}, \mathbf{v} \in V$ and $h, k \in K$ we have that

$$(h\mathbf{u} + k\mathbf{v})\alpha = h(\mathbf{u}\alpha) + k(\mathbf{v}\alpha). \tag{1.10}$$

The zero map, simply denoted by 0, is defined by $\mathbf{v}0 = \mathbf{0}$ for all $\mathbf{v} \in V$.

The idea of a linear map does not involve the way in which the vector spaces may be referred to a particular basis. However, in order to compute the image of individual vectors, it is usually necessary to choose bases for V and W. In this book we shall be concerned only with finite-dimensional vector spaces.

Let

$$\dim V = m, \qquad \dim W = n,$$

and write

$$V = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m], \quad W = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n] \tag{1.11}$$

to express that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are bases of V and W respectively.

The image of $\mathbf{p}_i$ under α is some vector in W and therefore a linear combination of the basis vectors of W. Thus we have a system of equations

$$\mathbf{p}_i\alpha = \sum_{j=1}^{n} a_{ij}\mathbf{q}_j \quad (i = 1, 2, \ldots, m), \tag{1.12}$$

where $a_{ij} \in K$. This information enables us to write down the image of any $\mathbf{v} \in V$ by what is known as the *principle of linearity*; for if

$$\mathbf{v} = \sum_{i=1}^{m} h_i\mathbf{p}_i,$$

the linearity property (1.10) implies that

$$\mathbf{v}\alpha = \sum_{i=1}^{m} h_i \mathbf{p}_i \alpha = \sum_{i=1}^{m} \sum_{j=1}^{n} h_i a_{ij} \mathbf{q}_j.$$

Hence we may state that the $m \times n$ matrix

$$A = (a_{ij})$$

describes the linear map α relative to the bases (1.11).

If we had used different bases, say

$$V = [\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_m], \qquad W = [\mathbf{q}'_1, \mathbf{q}'_2, \ldots, \mathbf{q}'_n], \tag{1.13}$$

the same linear map α would have been described by the matrix

$$B = (b_{\lambda\mu}),$$

whose coefficients appear in the equations

$$\mathbf{p}'_\lambda \alpha = \sum_{\mu=1}^{n} b_{\lambda\mu} \mathbf{q}'_\mu \quad (\lambda = 1, 2, \ldots, m). \tag{1.14}$$

The change of bases is expressed algebraically by equations of the form

$$\begin{aligned} \mathbf{p}_i &= \sum_{\lambda=1}^{m} t_{i\lambda} \mathbf{p}'_\lambda \quad (i = 1, 2, \ldots, m) \\ \mathbf{q}_j &= \sum_{\mu=1}^{n} s_{j\mu} \mathbf{q}'_\mu \quad (j = 1, 2, \ldots, n) \end{aligned} \tag{1.15}$$

where $T = (t_{i\lambda})$ and $S = (s_{j\mu})$ are non-singular (invertible) matrices of dimensions m and n respectively. Inverting the first set of equations we write

$$\mathbf{p}'_\lambda = \sum_{i=1}^{m} \tilde{t}_{\lambda i} \mathbf{p}_i \quad (\lambda = 1, 2, \ldots, m),$$

where $T^{-1} = (\tilde{t}_{\lambda i})$. The relationship between the matrices A and B can now be obtained as follows (for the sake of brevity we suppress the ranges of the summation suffixes):

$$\mathbf{p}'_\lambda \alpha = \sum_i \tilde{t}_{\lambda i} \mathbf{p}_i \alpha = \sum_{i,j} \tilde{t}_{\lambda i} a_{ij} \mathbf{q}_j = \sum_{i,j,\mu} \tilde{t}_{\lambda i} a_{ij} s_{j\mu} q'_\mu,$$

whence on comparing this result with (1.14) we have that

$$B = T^{-1} A S. \tag{1.16}$$

In the present context we are concerned with the situation in which $V=W$ and α is invertible. Such a linear map

$$\alpha : V \to V$$

is called an *automorphism* of V over K. The matrix which describes α relative to any basis is non-singular; and any two matrices A and B which express α relative to two different bases are connected by an equation of the form

$$B = T^{-1}AT. \tag{1.17}$$

The set of all automorphisms of V over K forms a group which we denote by

$$\mathscr{A}_K(V),$$

or simply by $\mathscr{A}(V)$, when the choice of the ground field can be taken for granted. If α_1 and α_2 are two elements of $\mathscr{A}(V)$, their product $\alpha_1\alpha_2$ is defined by operator composition, that is, if $\mathbf{v} \in V$, then

$$\mathbf{v}(\alpha_1\alpha_2) = (\mathbf{v}\alpha_1)\alpha_2.$$

We now consider representations of G by automorphisms of a vector space V. Thus we are interested in homomorphisms

$$G \to \mathscr{A}_K(V). \tag{1.18}$$

This means that with each element of x of G there is associated an automorphism

$$\alpha(x) : V \to V$$

in such a way that

$$\alpha(x)\alpha(y) = \alpha(xy) \quad (x, y \in G). \tag{1.19}$$

We call (1.18) an *automorphism representation* of G, with the understanding that a suitable vector space V over K is involved.

In order to compute $\alpha(x)$ we refer V to a particular basis, say

$$V = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m]. \tag{1.20}$$

Applying (1.12) to the case in which $V=W$ we find that the action of $\alpha(x)$ is described by a matrix

$$A(x) = (a_{ij}(x))$$

over K, where

$$\mathbf{p}_i\alpha(x) = \sum_{j=1}^{m} a_{ij}(x)\mathbf{p}_j \quad (i = 1, 2, \ldots, m). \tag{1.21}$$

By virtue of (1.19) the matrix function $A(x)$ satisfies

$$A(x)A(y) = A(xy).$$

When the basis of V is changed, $\alpha(x)$ is described by a matrix of the form

$$B(x) = T^{-1}A(x)T,$$

where T is a non-singular matrix over K which is independent of x. Thus a representation $\alpha(x)$ gives rise to a class of equivalent matrix representations $A(x), B(x), \ldots$. Conversely, if we start with a matrix representation $A(x)$ we can associate with it an automorphism representation $\alpha(x)$ by starting with an arbitrary vector space (1.20) and defining the action of $\alpha(x)$ by means of (1.21).

Summing up, we can state that the classes of equivalent matrix representations are in one-to-one correspondence with automorphism representations of suitable vector spaces.

It is advantageous to push abstraction one stage further. In an automorphism representation each element x of G is associated with an automorphism $\alpha(x)$ of V. We shall now denote this automorphism simply by x; in other words, we put

$$\mathbf{v}x = \mathbf{v}\alpha(x), \tag{1.22}$$

and we say that G acts on V in accordance with (1.22). Formally, this defines a right-hand multiplication of a vector in V by an element of G. It is convenient to make the following

Definition 1.2. *Let G be a group. The vector space V over K is called a G-module, if a multiplication $\mathbf{v}x$ ($\mathbf{v} \in V$, $x \in G$) is defined, subject to the rules:*

(*i*) $\mathbf{v}x \in V$;
(*ii*) $(h\mathbf{v}+k\mathbf{w})x = h(\mathbf{v}x)+k(\mathbf{w}x), \quad (\mathbf{v}, \mathbf{w} \in V; h, k \in K)$;
(*iii*) $\mathbf{v}(xy) = (\mathbf{v}x)y$;
(*iv*) $\mathbf{v}1 = \mathbf{v}$.

Let us verify that, in an abstract guise, this definition recaptures the notion of an automorphism representation. Indeed, (i) states that multiplication by x induces a map of V into itself; (ii) expresses that this map is linear; (iii) establishes the homomorphic property (1.19); finally, (iii) and (iv) imply that x and x^{-1} induce mutually inverse maps so that all these maps are invertible.

If V is a G-module, we say that V *affords* the automorphism representation defined in (1.22) or else the matrix representation $A(x)$ given by

(1.21), except that we now write $\mathbf{p}_i x$ instead of $\mathbf{p}_i \alpha(x)$, thus

$$\mathbf{p}_i x = \sum_{j=1}^{m} a_{ij}(x)\mathbf{p}_j \qquad (i = 1, 2, \ldots, m). \tag{1.23}$$

Representation theory can be expressed either in terms of matrices, or else in the more abstract language of modules. The foregoing discussion shows that the two methods are essentially equivalent. The matrix approach lends itself more readily to computation, while the use of modules tends to render the theory more elegant. We shall endeavour to keep both points of view before the reader's mind.

1.3. Characters

Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m. We consider the characteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{vmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \ldots & -a_{1m}(x) \\ -a_{21}(x) & \lambda - a_{22}(x) & \ldots & -a_{2m}(x) \\ \ldots & \ldots & & \ldots \\ -a_{m1}(x) & -a_{m2}(x) & \ldots & \lambda - a_{mm}(x) \end{vmatrix}.$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + \ldots + a_{mm}(x).$$

It is customary to call the right-hand side of this equation the *trace* of $A(x)$, abbreviated to tr $A(x)$, so that

$$\phi(x) = \operatorname{tr} A(x). \tag{1.24}$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the *character* of $A(x)$. If

$$B(x) = T^{-1}A(x)T \tag{1.25}$$

is a representation equivalent to $A(x)$, then

$$\det(\lambda I - B(x)) = \det(\lambda I - A(x)), \tag{1.26}$$

because

$$\lambda I - B(x) = T^{-1}(\lambda I - A(x))T,$$

whence (1.26) follows by taking determinants of each side. In particular, on comparing coefficients of λ^{m-1} in (1.26) we find that

$$b_{11}(x) + b_{22}(x) + \ldots + b_{mm}(x) = a_{11}(x) + a_{22}(x) + \ldots + a_{mm}(x),$$

that is, equivalent representations have the same character. Put in a different way, we can state that $\phi(x)$ expresses a property of the equivalence class of matrix representations of which $A(x)$ is a member; or again, $\phi(x)$ is associated with an automorphism representation of a suitable G-module. It is this invariant feature which makes the character a meaningful concept for our purpose.

Suppose that x and $y = t^{-1}xt$ are conjugate elements of G. Then in any matrix representation $A(x)$ we have that

$$A(y) = (A(t))^{-1}A(x)A(t).$$

On taking traces on both sides and identifying $A(t)$ with T in (1.25) we find that

$$\operatorname{tr} A(y) = \operatorname{tr} A(x),$$

that is, by (1.24), $\phi(x) = \phi(y)$. Thus, in every representation, the character is constant throughout each conjugacy class of G. Accordingly, we say that ϕ is a *class function* on G.

For reference, we collect our main results:

Proposition 1.1. *Let $A(x)$ be a matrix representation of G. Then the character*

$$\phi(x) = \operatorname{tr} A(x)$$

has the following properties:

(*i*) *equivalent representations have the same character;*
(*ii*) *if x and y are conjugate in G, then $\phi(x) = \phi(y)$.*

1.4. Reducibility

As often happens, we gain insight into a mathematical structure by studying 'subobjects'. This leads us to the distinction between reducible and irreducible representations.

Definition 1.3. *Let V be a G-module over K. We say that U is a submodule of V if*

(*i*) *U is a vector space (over K) contained in V, and*
(*ii*) *U is a G-module, that is $\mathbf{u}x \in U$ for all $\mathbf{u} \in U$ and $x \in G$.*

Every G-module V possesses the trivial submodules $U = V$ and $U = 0$. A non-trivial submodule is also called a *proper submodule*.

Definition 1.4. *A G-module is said to be reducible over K if it possesses a proper submodule; otherwise it is said to be irreducible over K.*