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Excerpt
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Classification and examples of integral equations

1.1 Introduction

The mechanics problem of calculating the time a particle takes to slide under gravity down a given smooth curve, from any point on the curve to its lower end, leads to an exercise in integration. The time, $f(Y)$ say, for the particle to descend from the height Y is given by an expression of the form

$$f(Y)=\int_0^Y \frac{\phi(y)\,dy}{(Y-y)^{\frac{1}{2}}} \quad (0\leqslant Y\leqslant b), \tag{1.1}$$

where $\phi(y)$ embodies the shape of the given curve.

The converse problem, in which the time of descent from height Y is given and the particular curve which produces this time has to be found, is less straightforward, as it entails the determination of the function ϕ from (1.1), $f(Y)$ now being assigned for $0\leqslant Y\leqslant b$. From this point of view, (1.1) is called an integral equation, this description expressing the fact that the function to be determined appears under an integral sign. The equation (1.1) is one of historical importance, attributed to Abel.

Many readers will have already encountered integral equations, but perhaps only in a context where this terminology is not used. For example, the pair of equations

$$\left.\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt}\phi(t)\,dt=f(x) \quad (-\infty<x<\infty),\\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixt}f(t)\,dt=\phi(x) \quad (-\infty<x<\infty), \end{aligned}\right\} \tag{1.2}$$

which defines the Fourier transform and its inverse, may be viewed as an integral equation and its solution. If $f(x)$ is regarded as known for $-\infty<x<\infty$ in the first equation, then the solution for $\phi(x)$ is provided by the second equation, also for $-\infty<x<\infty$.

Our illustration (1.1) from particle mechanics is one of many integral equations which result directly from a physical problem. Other problems, in elasticity theory and fluid mechanics for example, whose natural formulations are in terms of differential equations also provide a plentiful supply of integral equations. This is because initial and boundary value problems for differential equations can often be converted into integral equations and there are usually significant advantages to be gained from making use of this conversion. The intimate relationship between differential and integral equations forms the basis of a theme running through the book and is used to provide many of the illustrations. It allows us to present neutral examples, removed from the particular application areas where they originated and accessible to all readers, although we shall outline the context from which a problem has been drawn where this is appropriate.

Before deriving some examples of integral equations, it is convenient to introduce the principal features of their classification, so that the different types can be identified as they arise. We do this in the next section and follow it in §1.3 by providing illustrations of those varieties of integral equation with which we shall be concerned in subsequent chapters. We also take the opportunity to introduce at an early stage some ideas which can be used in the development of the general theory. The reader not familiar with partial differential equations can omit the details given in §1.3.3. This is the only section in which a knowledge of partial differential equations is needed.

1.2 Classification of integral equations

We are concerned, for the most part, with integral equations in which integration is with respect to a single real variable. The extension of the terminology and methods to higher order integral equations, where these do appear, is straightforward.

The notation adopted in this section, and throughout much of the text, is as follows. The unknown function will be denoted by ϕ or $\phi(x)$. Every integral equation contains a function obtained from ϕ by integration and of the form $\int_a^b k(x,t)\phi(t)\,\mathrm{d}t$, where k is called the *kernel* and is assumed known. For example, in the integral equation

$$\phi(x)=\int_0^1 |x-t|\phi(t)\,\mathrm{d}t+f(x) \quad (0\leqslant x\leqslant 1)$$

the kernel is given by $k(x,t)=|x-t|$, and the function f, called the *free term*, is also assumed known. In general the kernel and free term will be complex-valued functions of real variables. A condition such as $(0\leqslant x\leqslant 1)$

following an equation indicates that the equation holds for all values of x in the given interval. Thus for the integral equation given above, we seek a solution $\phi(x)$ satisfying the equation for all x in $[0, 1]$.

For definiteness we shall assume for the present that each integral equation holds in a closed interval. In fact some equations encountered later may hold only in open, or half-open, intervals. Analogous use of conditions like $(0 \leqslant x \leqslant 1)$ will be made in other circumstances.

The classification of integral equations centres on three basic characteristics which together describe their overall structure, and it is useful to set these down briefly before entering into greater detail.

(i) The *kind* of an equation refers to the location of the unknown function. *First* kind equations have the unknown function present under the integral sign only; *second* and *third* kind equations also have the unknown function outside the integral.

(ii) The historical descriptions *Fredholm* and *Volterra* are concerned with the integration interval. In a Fredholm equation the integral is over a finite interval with fixed end-points; in a Volterra equation the integral is indefinite.

(iii) The adjective *singular* is sometimes used when the integration is improper, either because the interval is infinite, or because the integrand is unbounded within the given interval. Obviously an integral equation can be singular on both counts.

Fredholm equations are therefore distinguished by having fixed, finite limits of integration. We denote these limits by a and b here, but we shall usually take $a = 0$ and $b = 1$ later, noting that the interval $[0, 1]$ can be transformed to a general finite interval $[a, b]$ by a simple change of variable. The Fredholm equation of the first kind is

$$f(x) = \int_a^b k(x, t)\phi(t)\,dt \quad (a \leqslant x \leqslant b), \tag{1.3}$$

and the Fredholm equation of the second kind is

$$\phi(x) = f(x) + \lambda \int_a^b k(x, t)\phi(t)\,dt \quad (a \leqslant x \leqslant b). \tag{1.4}$$

The quantity appearing in (1.4) which we have not mentioned so far, λ, is a numerical parameter, generally complex. It plays a crucial part in the theory of (1.4); in practical applications λ is usually composed of physical quantities and our later examples will show why it is not simply absorbed into the kernel.

Equation (1.4) is the standard form of the second kind Fredholm equation, but an alternative version which sometimes proves to be more convenient to deal with is

$$\mu\phi(x) = f(x) + \int_a^b k(x,t)\phi(t)\,dt \quad (a \leqslant x \leqslant b). \tag{1.5}$$

In this representation, the parameter μ takes over the role of λ, the two being related simply by

$$\lambda\mu = 1. \tag{1.6}$$

We are merely observing here that λ can be divided through (1.4) (the case $\lambda = 0$ can be disregarded as it is trivial) and (1.6) used to replace $1/\lambda$ by μ; μ can be absorbed into the given free term f without any loss of generality. There are advantages in each of the versions (1.4) and (1.5) and for this reason we retain both λ and μ, using whichever is the more convenient at any stage; the two symbols will have no other uses and will always be related by (1.6). One obvious advantage of the form (1.5) is that setting $\mu = 0$ reduces it to the first kind equation (1.3), apart from a sign change which can be easily accommodated in the free term f.

The Fredholm equation of the third kind,

$$\mu\psi(x)\phi(x) = f(x) + \int_a^b k(x,t)\phi(t)\,dt \quad (a \leqslant x \leqslant b), \tag{1.7}$$

where ψ is an assigned function, may be regarded as a generalisation of (1.5). If ψ does not vanish in $[a, b]$, it can be divided out and absorbed into f and k. For this reason (1.7) is of limited practical significance.

We note that the specification of the interval in which the integral equation holds is an inherent part of the statement of the problem. For the first kind equation (1.3) it is not necessary that this interval should coincide with the integration interval, although it usually does.

Volterra equations differ from Fredholm equations, as we have already noted, in that the integration is indefinite. Retaining the understanding that a and b are fixed and finite, the classical form of the first kind Volterra equation is

$$f(x) = \int_a^x k(x,t)\phi(t)\,dt \quad (a \leqslant x \leqslant b), \tag{1.8}$$

and the corresponding second kind Volterra equation can be written as

$$\phi(x)=f(x)+\lambda\int_a^x k(x,t)\phi(t)\,dt \quad (a\leqslant x\leqslant b), \tag{1.9}$$

or in the alternative form

$$\mu\phi(x)=f(x)+\int_a^x k(x,t)\phi(t)\,dt \quad (a\leqslant x\leqslant b). \tag{1.10}$$

The third kind equation has the given multiplier $\psi(x)$ present on the left of (1.9) or (1.10).

There is obviously ample scope for variations on these standard forms. The integration interval can be altered to give, for example, the first kind equation

$$f(x)=\int_x^b k(x,t)\phi(t)\,dt \quad (a\leqslant x\leqslant b), \tag{1.11}$$

which will prove to be of some importance in due course. Other generalisations of (1.8) and (1.9) or (1.10) can be considered.

One further item of terminology associated with both Fredholm and Volterra equations is that they are said to be *homogeneous* if $f(x)=0$ in $[a, b]$, and *inhomogeneous* otherwise. In effect we have previously set down the inhomogeneous versions of the various equations and, for example, either of the forms

$$\left.\begin{aligned}\phi(x)&=\lambda\int_a^b k(x,t)\phi(t)\,dt\\ \mu\phi(x)&=\int_a^b k(x,t)\phi(t)\,dt\end{aligned}\right\}\quad (a\leqslant x\leqslant b), \tag{1.12}$$

is referred to as the homogeneous Fredholm equation of the second kind.

It must be remarked here that one property of the integral equations we have introduced is that the principle of superposition applies to their homogeneous versions. That is, if ϕ_1 and ϕ_2 are both solutions of (1.12) for given values of λ and μ, then the function $c_1\phi_1+c_2\phi_2$ is also a solution for any constants c_1 and c_2. Thus the condition defining a *linear* problem is met, Fredholm and Volterra equations being examples of linear integral equations.

We should also note that Fredholm equations reduce to those of Volterra type if their kernels are defined to have the property that

$$k(x,t)=0 \quad (a\leqslant x\leqslant t\leqslant b). \tag{1.13}$$

This relationship between the two varieties of equation is a useful one, but it is wrong to infer that the differences between them are minimal.

The third feature of the principal classification given earlier relates to improper integrals, which occur widely in application areas. Strictly speaking, an integral equation should be described as singular if

(a) at least one of the integration limits, or the interval in which the equation holds, is infinite,

or (b) the kernel is unbounded in the given interval.

It turns out, however, that rigid adherence to this terminology is both unnecessary and misleading.

As the definitions of Fredholm and Volterra equations specifically excluded infinite integrals, case (a) does provide us with a new type of equation which includes the Fourier transform pair (1.2). On the other hand we made no stipulations regarding the kernels of Fredholm and Volterra equations. Whilst it is premature to attempt to do this here, we can anticipate the analysis to the extent of remarking that certain types of unboundedness in kernels are permitted in our general theory of Fredholm and Volterra equations. In such cases we do not need to distinguish the equations by the addition of the adjective 'singular' to the usual Fredholm or Volterra description, although we may choose to do so for emphasis.

The so-called *weakly singular* kernel

$$k(x, t)=\frac{\hat{k}(x, t)}{|x-t|^{\alpha}}, \tag{1.14}$$

where $\alpha \in (0, 1)$ is given and $\hat{k}$ is a bounded function, is an example of an unbounded kernel which needs no special treatment. However, we shall frequently attach the description 'weakly singular' to an integral equation with a kernel of the form (1.14), because of the prominent place occupied by such equations in application areas. It should be noted in particular that the equally important logarithmically singular kernel

$$k(x, t)=\hat{k}(x, t)\log|x-t|, \tag{1.15}$$

where $\hat{k}$ is bounded, may be regarded as weakly singular, as it can be written in the form

$$k(x, t)=\frac{\hat{k}(x, t)|x-t|^{\epsilon}\log|x-t|}{|x-t|^{\epsilon}},$$

the numerator of which is bounded for any $\epsilon>0$.

Our first example, (1.1), of an integral equation has a kernel of the form

(1.14) with $\alpha = \frac{1}{2}$ and its full description is as a weakly singular Volterra equation of the first kind.

Special consideration does have to be given to what are often termed *strongly singular* or *Cauchy singular* kernels. These are typified by

$$k(x, t) = \frac{\hat{k}(x, t)}{x - t}, \tag{1.16}$$

where $\hat{k}$ is bounded. Within the framework we develop these are the only truly singular kernels of practical interest. We shall, however, allow a less restrictive condition on $\hat{k}$ than that given here, as regards (1.14) and (1.16).

We have now covered the principal elements arising in the classification of integral equations. There are of course items of subsidiary terminology which we do not introduce until the need arises.

1.3 A collection of examples

Here we give particular illustrations of the integral equation types introduced in the previous section, drawing on some straightforward problems which can be dealt with in an *ad hoc* manner. We also use these examples to expose some features of integral equations which can be pursued more fully in later chapters.

1.3.1 Initial value problems for ordinary differential equations

Suppose that ϕ satisfies

$$\left.\begin{aligned} &\phi'(x) = F(x, \phi(x)) \quad (0 < x < 1), \\ &\phi(0) = \phi_0, \end{aligned}\right\} \tag{1.17}$$

where the function F and the number ϕ_0 are given. We assume that ϕ is continuous in the closed interval $[0, 1]$ which, in particular, allows the initial condition to be sensibly interpreted. Integration then gives

$$\phi(x) = \int_0^x F(t, \phi(t))\, \mathrm{d}t + \phi_0 \quad (0 \leqslant x \leqslant 1). \tag{1.18}$$

Conversely if ϕ is a continuous function satisfying (1.18) then $\phi(0) = \phi_0$ and the integral may be differentiated to give (1.17). Therefore, provided that all the functions are sufficiently well-behaved that the integration and differentiation may be performed, (1.17) and (1.18) have the same solutions and are in this sense equivalent. (Although we do not wish to become involved with the analysis at this stage, it is worth noting that if F is continuous then a continuous function ϕ which satisfies (1.17) also satisfies (1.18) and a solution of (1.18) is continuous and satisfies (1.17).)

We can proceed in the same way for the second order initial value problem

$$\left.\begin{aligned}&\phi''(x)=F(x,\phi(x)) \quad (0<x<1),\\ &\phi(0)=\phi_0, \quad \phi'(0)=\phi'_0,\end{aligned}\right\} \tag{1.19}$$

where the number ϕ'_0 is additionally assigned. Again we must make a stipulation regarding continuity, and to avoid the need to raise this issue repeatedly we adopt the convention that, unless otherwise indicated, in a problem such as (1.19) ϕ and its derivatives up to the highest order specified at end points of the interval are extended to continuous functions on the closed interval.

One integration gives

$$\phi'(x)=\int_0^x F(t,\phi(t))\,\mathrm{d}t+\phi'_0 \quad (0\leqslant x\leqslant 1),$$

satisfying $\phi'(0)=\phi'_0$ and a second produces

$$\phi(x)=\int_0^x \mathrm{d}s\int_0^s F(t,\phi(t))\,\mathrm{d}t+\phi'_0x+\phi_0 \quad (0\leqslant x\leqslant 1), \tag{1.20}$$

the further constant of integration having been chosen so that $\phi(0)=\phi_0$.

Simplification of the repeated integral in (1.20) follows on using the result

$$\int_0^x \mathrm{d}s\int_0^s G(s,t)\,\mathrm{d}t=\int_0^x \mathrm{d}t\int_t^x G(s,t)\,\mathrm{d}s, \tag{1.21}$$

for which it is sufficient that G be a continuous function of both variables. To establish (1.21) note that the repeated integral on the left hand side is evaluated over the shaded triangular region of the t–s plane shown in Figure 1.1. The inner integral is evaluated at a fixed s from $t=0$ to $t=s$ and the outer integral then runs from $s=0$ to $s=x$. On reversing the integration order the same triangular region must be covered. This is achieved by integrating from $s=t$ to $s=x$ at a fixed t, followed by integration with respect to t from $t=0$ to $t=x$.

If we assume that F is a continuous function of both variables, then (1.21) gives

$$\int_0^x \mathrm{d}s\int_0^s F(t,\phi(t))\,\mathrm{d}t=\int_0^x (x-t)F(t,\phi(t))\,\mathrm{d}t,$$

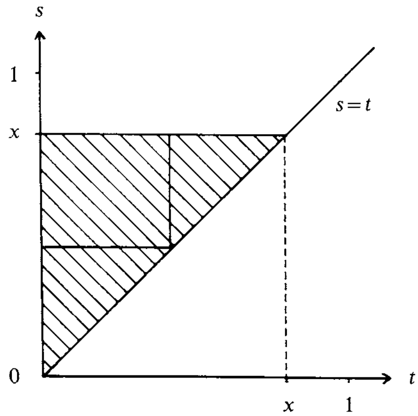

Figure 1.1

and the integral equation corresponding to (1.19) in its simplest form is

$$\phi(x)=\int_0^x (x-t)F(t,\phi(t))\,dt+\phi_0'x+\phi_0, \quad (0\leqslant x\leqslant 1). \tag{1.22}$$

If ϕ is a continuous solution of (1.22) then differentiation under the integral sign shows that ϕ also satisfies (1.19) and so the two problems for ϕ correspond.

So far the formulation has allowed for non-linear equations. Clearly, if $F(x, y)$ is linear in y, then the integral equation (1.22) is also linear and in this case we can describe it as an inhomogeneous second kind Volterra equation.

Example 1.1

Let $\phi(x)$ satisfy Airy's equation

$$\phi''(x)=x\phi(x) \quad (0<x<1), \tag{1.23}$$

with

$$\phi(0)=1, \quad \phi'(0)=0.$$

Putting $F(x, y)=xy$, $\phi_0=1$ and $\phi_0'=0$ in (1.22) gives the integral equation

$$\phi(x)=\int_0^x (x-t)t\phi(t)\,dt+1 \quad (0\leqslant x\leqslant 1), \tag{1.24}$$

the solution of which is also that solution of (1.23) which satisfies the prescribed initial conditions. □

We can conduct a preliminary investigation of second kind Volterra equations at this point, using (1.24) for illustration. In the interests of algebraic simplicity, suppose that we introduce the operator K by

$$(K\phi)(x)=\int_0^x (x-t)t\phi(t)\,dt \quad (0\leqslant x\leqslant 1) \tag{1.25}$$

and let $f(x)=1$. Then, for example,

$$(Kf)(x)=\int_0^x (x-t)t\,dt=[\tfrac{1}{2}xt^2-\tfrac{1}{3}t^3]_{t=0}^{t=x}=\tfrac{1}{6}x^3 \quad (0\leqslant x\leqslant 1). \tag{1.26}$$

Although our use of K at this stage is merely as a shorthand, we can anticipate our later central viewpoint by noting that K defines a linear mapping of functions into functions.

The integral equation (1.24) can now be written in the abbreviated form

$$\phi(x)=f(x)+(K\phi)(x) \quad (0\leqslant x\leqslant 1),$$

or even more briefly as

$$\phi=f+K\phi, \tag{1.27}$$

where the notation assumes that the interval in which the equation holds is prescribed; in this case the interval is, of course, $[0, 1]$. If we substitute the expression for ϕ given by the right hand side of (1.27) into the term $K\phi$ of (1.27), we obtain

$$\phi=f+K(f+K\phi),$$

that is

$$\phi=f+Kf+K^2\phi, \tag{1.28}$$

where $K^2\phi=K(K\phi)$ denotes that the operator K has to be applied to the function ϕ twice in succession. Referring to (1.26) we have, for example,

$$(K^2f)(x)=\int_0^x (x-t)t(Kf)(t)\,dt$$

$$=\int_0^x (x-t)t(\tfrac{1}{6}t^3)\,dt=\tfrac{1}{180}x^6 \quad (0\leqslant x\leqslant 1). \tag{1.29}$$

Substituting (1.27) into the right hand side of (1.28) results in

$$\phi=f+Kf+K^2f+K^3\phi, \tag{1.30}$$

which we can make more explicit by putting $f=1$ and using (1.26) and