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CONTENTS

Foreword	ix
Introduction	1
1 Inner product spaces	4
1.1 Inner product spaces as metric spaces	6
1.2 Problems	11
2 Normed spaces	13
2.1 Closed linear subspaces	15
2.2 Problems	18
3 Hilbert and Banach spaces	21
3.1 The space $L^2(a, b)$	23
3.2 The closest point property	26
3.3 Problems	28
4 Orthogonal expansions	31
4.1 Bessel's inequality	34
4.2 Pointwise and L^2 convergence	35
4.3 Complete orthonormal sequences	36
4.4 Orthogonal complements	39
4.5 Problems	42
5 Classical Fourier series	45
5.1 The Fejér kernel	46
5.2 Fejér's theorem	52
5.3 Parseval's formula	54
5.4 Weierstrass' approximation theorem	54
5.5 Problems	55
6 Dual spaces	59
6.1 The Riesz–Fréchet theorem	62
6.2 Problems	64

7 Linear operators	67
7.1 The Banach space $\mathcal{L}(E, F)$	71
7.2 Inverses of operators	72
7.3 Adjoint operators	75
7.4 Hermitian operators	78
7.5 The spectrum	80
7.6 Infinite matrices	82
7.7 Problems	83
8 Compact operators	89
8.1 Hilbert–Schmidt operators	92
8.2 The spectral theorem for compact Hermitian operators	96
8.3 Problems	102
9 Sturm–Liouville systems	105
9.1 Small oscillations of a hanging chain	105
9.2 Eigenfunctions and eigenvalues	111
9.3 Orthogonality of eigenfunctions	114
9.4 Problems	115
10 Green’s functions	119
10.1 Compactness of the inverse of a Sturm–Liouville operator	124
10.2 Problems	128
11 Eigenfunction expansions	131
11.1 Solution of the hanging chain problem	134
11.2 Problems	138
12 Positive operators and contractions	141
12.1 Operator matrices	144
12.2 Möbius transformations	146
12.3 Completing matrix contractions	149
12.4 Problems	152
13 Hardy spaces	157
13.1 Poisson’s kernel	161
13.2 Fatou’s theorem	164
13.3 Zero sets of H^2 functions	169
13.4 Multiplication operators and infinite Toeplitz and Hankel matrices	171
13.5 Problems	174
14 Interlude: complex analysis and operators in engineering	177
15 Approximation by analytic functions	187
15.1 The Nehari problem	189

15.2	Hankel operators	190
15.3	Solution of Nehari's problem	196
15.4	Problems	200
16	Approximation by meromorphic functions	203
16.1	The singular values of an operator	204
16.2	Schmidt pairs and singular vectors	206
16.3	The Adamyan–Arov–Krein theorem	210
16.4	Problems	219
	<i>Appendix: square roots of positive operators</i>	221
	<i>References</i>	225
	<i>Answers to selected problems</i>	226
	<i>Afterword</i>	230
	Index of notation	236
	Subject index	238

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FOREWORD

The basic notions of the theory of Hilbert space are current in many parts of pure and applied mathematics, and in physics, engineering and statistics. They are well worth a place in any honours mathematics course, and Chapters 1 to 8 of this book aim to present them in a way accessible to undergraduate students. A course in Hilbert space is likely to be the last analysis course for many students, and it should therefore be able to stand on its own: it should not depend for its motivation on further study of abstract analysis, but should as far as possible have a value which is apparent either on aesthetic grounds or for its scientific or practical applications. For this reason I have included more historical and background material than is customary, and have omitted some of the major theorems about Banach spaces which are traditionally taught in introductory courses on functional analysis, but which are really more appropriate to students who will be pursuing operator theory further (the closed graph, Hahn–Banach and uniform boundedness theorems). The second half of the book describes two substantial applications. One of these is standard: the Sturm–Liouville theory of eigenfunction expansions, and its role in the solution of the partial differential equations of mathematical physics by the method of separation of variables. The other (in Chapters 12 to 16) is less common, but is nevertheless ideal for a final year course. It is beautiful mathematics, it is relatively recent and visibly useful. It also entails the development of some standard operator theory along the way, and exhibits very well the connection between abstract analysis and the more classical field of complex analysis.

Although the book was written primarily for an undergraduate audience, I hope it may be found useful for graduate courses also. I firmly believe that functional analysis is best approached through a sound grounding in Hilbert space theory, and am confident that students will be

better able to benefit from one of the many excellent advanced texts on functional analysis and its applications if they first master the material contained herein. Chapters 12 to 16 may also be of interest to some electrical engineers. Some recent developments, particularly in control and filter design, require familiarity with this aspect of operator theory.

Chapters 1 to 8 are based on a compulsory course of twenty lectures which I gave to third year honours students at Glasgow University, and the remainder of the book, with a few omissions, on an optional twenty lecture course for fourth year students. In forty lectures at the undergraduate level it should be possible to cover the whole book except for the Adamyán–Arov–Krein theorem and the proofs of Fatou’s theorem and the existence of square roots of positive operators. Chapters 12 to 16 do not depend on Chapters 9 to 11: they can be read straight after Chapter 8.

The book presupposes introductory courses in real analysis, linear algebra and topology (metric spaces suffice). For Chapters 12 to 16, and some of the problems earlier in the book, elementary complex analysis is required. It is tacitly assumed in Chapters 9 to 11 that the reader has met differential equations before, though formal requirements are slight. I have taken pains *not* to assume knowledge of the Lebesgue integral: the reader is asked only to believe that there is a definition of integral which makes $L^2(a, b)$ complete and the continuous functions a dense subspace. However, I am obliged to admit that there are parts of Chapter 13 which will feel distinctly more comfortable to those who are familiar with Lebesgue measure.

I am grateful to Dr Frances Goldman and Dr Philip Spain for reading the text and making useful suggestions. I am also very thankful that, despite the all-conquering march of the word processor, Cambridge University Press was willing to accept manuscript, so that I do not have to thank anyone for his excellent typing.