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SPINORS AND SPACE–TIME

Volume 1: Two-spinor calculus and relativistic fields

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Two-spinor calculus and relativistic fields

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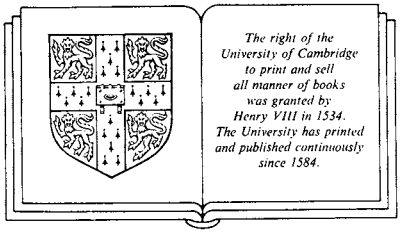

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Preface

To a very high degree of accuracy, the space–time we inhabit can be taken to be a smooth four-dimensional manifold, endowed with the smooth Lorentzian metric of Einstein's special or general relativity. The formalism most commonly used for the mathematical treatment of manifolds and their metrics is, of course, the tensor calculus (or such essentially equivalent alternatives as Cartan's calculus of moving frames). But in the specific case of four dimensions and Lorentzian metric there happens to exist – by accident or providence – another formalism which is in many ways more appropriate, and that is the formalism of 2-spinors. Yet 2-spinor calculus is still comparatively unfamiliar even now – some seventy years after Cartan first introduced the general spinor concept, and over fifty years since Dirac, in his equation for the electron, revealed a fundamentally important role for spinors in relativistic physics and van der Waerden provided the basic 2-spinor algebra and notation.

The present work was written in the hope of giving greater currency to these ideas. We develop the 2-spinor calculus in considerable detail, assuming no prior knowledge of the subject, and show how it may be viewed either as a useful supplement or as a practical alternative to the more familiar world-tensor calculus. We shall concentrate, here, entirely on 2-spinors, rather than the 4-spinors that have become the more familiar tools of theoretical physicists. The reason for this is that only with 2-spinors does one obtain a practical alternative to the standard vector–tensor calculus, 2-spinors being the more primitive elements out of which 4-spinors (as well as world-tensors) can be readily built.

Spinor calculus may be regarded as applying at a deeper level of structure of space–time than that described by the standard world-tensor calculus. By comparison, world-tensors are less refined, fail to make transparent some of the subtler properties of space–time brought particularly to light by quantum mechanics and, not least, make certain types of mathematical calculations inordinately heavy. (*Their* strength lies in a general applicability to manifolds of arbitrary dimension, rather than in supplying a specific space–time calculus.)

In fact any world-tensor calculation can, by an obvious prescription, be translated entirely into a 2-spinor form. The reverse is also, in a sense, true – and we shall give a comprehensive treatment of such translations later in this book – though the tensor translations of simple spinor manipulations can turn out to be extremely complicated. This effective equivalence may have led some 'sceptics' to believe that spinors are 'unnecessary'. We hope that this book will help to convince the reader that there are many classes of spinorial results about space–time which would have lain undiscovered if only tensor methods had been available, and others whose antecedents and interrelations would be totally obscured by tensor descriptions.

When appropriately viewed, the 2-spinor calculus is also simpler than that of world-tensors. The essential reason is that the basic spin-space is two-complex-dimensional rather than four-real-dimensional. Not only are two dimensions easier to handle than four, but complex algebra and complex geometry have many simple, elegant and uniform properties not possessed by their real counterparts.

Additionally, spinors seem to have profound links with the complex numbers that appear in quantum mechanics.* Though in this work we shall not be concerned with quantum mechanics as such, many of the techniques we describe are in fact extremely valuable in a quantum context. While our discussion will be given entirely classically, the formalism can, without essential difficulty, be adapted to quantum (or quantum-field-theoretic) problems.

As far as we are aware, this book is the first to present a comprehensive development of space–time geometry using the 2-spinor formalism. There are also several other new features in our presentation. One of these is the systematic and consistent use of the *abstract index* approach to tensor and spinor calculus. We hope that the purist differential geometer who casually leafs through the book will not automatically be put off by the appearance of numerous indices. Except for the occasional bold-face upright ones, our indices differ from the more usual ones in being abstract markers without reference to any basis or coordinate system. Our use of abstract indices leads to a number of simplifications over conventional treatments. The use of some sort of index notation seems, indeed, to be virtually essential in order that the necessary detailed manipulations can

* The view that space–time geometry, as well as quantum theory, may be governed by an underlying complex rather than real structure is further developed in the theory of twistors, which is just one of the several topics discussed in the companion volume to the present work: *Spinors and space–time, Vol. 2: Spinor and twistor methods in space–time geometry*, (Cambridge University Press 1985).

be presented in a transparent form. (In an appendix we outline an alternative and equivalent diagrammatic notation which is very valuable for use in private calculations.)

This book appears also to be breaking some new ground in its presentation of several other topics. We provide explicit geometric realizations not only of 2-spinors themselves but also of their various algebraic operations and some of the related topology. We give a host of useful lemmas for both spinor and general tensor algebra. We provide the first comprehensive treatment of (not necessarily normalized) spin-coefficients which includes the compacted spin- and boost-weighted operators $\eth$ and þ and their conformally invariant modifications $\eth_{\mathscr{C}}$ and $\text{þ}_{\mathscr{C}}$. We present a general treatment of conformal invariance; and also an abstract-index-operator approach to the electromagnetic and Yang–Mills fields (in which the somewhat ungainly appearance of the latter is, we hope, compensated by the comprehensiveness of our scheme). Our spinorial treatment of (spin-weighted) spherical harmonics we believe to be new. Our presentation of exact sets of fields as the systems which propagate uniquely away from arbitrarily chosen null-data on a light cone has not previously appeared in book form; nor has the related explicit integral spinor formula (the generalized Kirchhoff–d'Adhémar expression) for representing massless free fields in terms of such data. The development we give for the interacting Maxwell–Dirac theory in terms of sums of integrals described by zig-zag and forked null paths appears here for the first time.

As for the genesis of this work, it goes back to the spring of 1962 when one of us (R.P.) gave a series of seminars on the then-emerging subject of 2-spinors in relativity, and the other (W.R.) took notes and became more and more convinced that these notes might usefully become a book. A duplicated draft of the early chapters was distributed to colleagues that summer. Our efforts on successive drafts have waxed and waned over the succeeding years as the subject grew and grew. Finally during the last three years we made a concerted effort and re-wrote and almost doubled the entire work, and hope to have brought it fully up to date. In its style we have tried to preserve the somewhat informal and unhurried manner of the original seminars, clearly stating our motivations, not shunning heuristic justifications of some of the mathematical results that are needed, and occasionally going off on tangents or indulging in asides. There exist many more rapid and condensed ways of arriving at the required formalisms, but we preferred a more leisurely pace, partly to facilitate the progress of students working on their own, and partly to underline the down-to-earth utility of the subject.

Fortunately our rather lengthy manuscript allowed a natural division into two volumes, which can now be read independently. The essential content of Vol. 1 is summarized in an introductory section to Vol. 2. References in Vol. 1 to Chapters 6–9 refer to Vol. 2.

We owe our thanks to a great many people. Those whom we mention are the ones whose specific contributions have come most readily to mind, and it is inevitable that in the period of over twenty years in which we have been engaged in writing this work, some names will have escaped our memories. For a variety of different kinds of assistance we thank Nikos Batakis, Klaus Bichteler, Raoul Bott, Nick Buchdahl, Subrahmanyan Chandrasekhar, Jürgen Ehlers, Leon Ehrenpreis, Robert Geroch, Stephen Hawking, Alan Held, Nigel Hitchin, Jim Isenberg, Ben Jeffryes, Saunders Mac Lane, Ted Newman, Don Page, Felix Pirani, Ivor Robinson, Ray Sachs, Engelbert Schücking, William Shaw, Takeshi Shirafuji, Peter Szekeres, Paul Tod, Nick Woodhouse, and particularly, Dennis Sciama for his continued and unfailing encouragement. Our thanks go also to Markus Fierz for a remark leading to the footnote on p. 321. Especially warm thanks go to Judith Daniels for her encouragement and detailed criticisms of the manuscript when the writing was going through a difficult period. We are also greatly indebted to Tsou Sheung Tsun for her caring assistance with the references and related matters. Finally, to those people whose contributions we can no longer quite recall we offer both our thanks and our apologies.

Roger Penrose
Wolfgang Rindler

1984