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# *Solitons: an Introduction*

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To Judith and Rosalind

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# *Preface*

The theory of solitons is attractive and exciting; it brings together many branches of mathematics, some of which touch on deep ideas. Several of its aspects are amazing and beautiful; we shall present some of them in this book. The theory is, nevertheless, related to even more areas of mathematics, and has even more applications to the physical sciences, than the number which are included here. It has an interesting history and a promising future. Indeed, the work of Kruskal and his associates which gave us the 'inverse scattering transform' – a grand title for soliton theory – is a major achievement of twentieth-century mathematics. Their work was stimulated by a physical problem together with some surprising computational results. This is a classic example of how numerical results lead to the development of new mathematics, just as observational and experimental results have done since the time of Archimedes.

This book has grown out of *Solitons* written by one of us (PGD). That book originated from lectures given to final-year mathematics honours students at the University of Bristol. Much of the material in this version has also been used as the basis for an introductory course on inverse scattering theory given to MSc students at the University of Newcastle upon Tyne. In both courses the aim was to present the essence of inverse scattering clearly, rather than to develop the theory rigorously and completely. That is also the overall aim of this book. It is intended for senior undergraduate students, and postgraduate students, in physics, chemistry, and engineering, as well as mathematics. The book will also help specialists in these and other fields to learn the theory of solitons. However, the theory is not taken as far as the rapidly advancing frontiers of research.

This book introduces the fundamental ideas underlying the inverse scattering transform from the point of view of a course of advanced calculus or the methods of mathematical physics. Some knowledge of the elements of the theories of linear waves, partial differential equations, Fourier integrals, the calculus of variations, Sturm–Liouville theory and the hypergeometric function, but little more, is assumed. Also, some familiarity with the main ingredients of the theories of water waves, continuous groups, elliptic functions and Hilbert spaces will be useful, but is not essential. The relevant ideas from one-dimensional wave mechanics (both scattering and inverse scattering), necessary for the presentation of the inverse scattering transform, are described. References are given in the text (or at the end of each chapter) to help readers to learn more of the foregoing topics. Some of the diverse applications of the theory of solitons are mentioned only briefly, either in the main text, or in the exercises at the end of each chapter. However, the Korteweg–de Vries equation is derived for a water-wave problem.

The material is presented as simply as we can, and a number of worked examples are also used to help the reader follow the various ideas. Of course, some parts of the theory are more exacting than others, and some problems are more difficult than others. The more difficult sections, paragraphs and set problems are indicated by asterisks; these passages may be omitted on a first reading of the book. Further reading is offered at the end of each chapter to direct the reader to more detailed treatments of some of the topics. The sections are numbered according to the decimal system, and the equations are numbered according to the chapter in which they appear, e.g. equation (1.2) is equation 2 of Chap. 1. The exercises are similarly numbered (e.g. Q1.2), as are the answers (e.g. A1.2) at the end of the book.

We are grateful to Miss Sarah Trickett (Figs. 4.5, 4.7, 4.8), Mr Mark Lewy (Fig. 8.1), Dr Adam Wheeler (Fig. 8.2), Mr Gregory Jones (Figs. 8.3, 8.4, 8.6, 8.7) and Dr Stephen Thompson (Figs. 8.8, 8.9) for their computations and plots of solutions on which our figures have been based; to Miss Carolyn Pharoah and Miss Alison Davies for their clear draughtsmanship of the figures; to Professor Neil Freeman (various points) and Dr Andrew Wathen (§7.3) for technical advice; to Academic Press (copyright of Figs. 8.8, 8.9); and to Mrs Heather Bliss, Mrs Hilary Cartwright and Mrs Nancy Thorp for their careful and cheerful typing of the text.

We have corrected some misprints and errors in the 1990 reprint and in the present one. We are grateful to Prof. P. S. Landweber for suggesting several of these improvements.

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