1 The Korteweg–de Vries equation

1.1 Preliminaries

Wave phenomena abound in mathematical physics, and are met early in undergraduate courses. They may be first introduced as waves on a string, or perhaps on the surface of water, or in a stretched membrane. With a little more background they may be discussed in connection with sound – and then shock waves may be mentioned, or (for the physics student) the first meeting might be via electromagnetic waves. In all these areas it is common practice to develop the concepts of wave propagation from the simplest – albeit idealised – model for one-dimensional motion,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $u(x, t)$ is the amplitude of the wave and c is a positive constant. This equation has a simple and well-known general solution, expressed in terms of *characteristic variables* $(x \pm ct)$ as

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1.2}$$

where f and g are arbitrary functions. (In keeping with the usual convention, we shall regard t as a time coordinate and x as a spatial coordinate, although in equation (1.1) these are readily interchangeable since they differ only by the 'scaling' factor, c.)

The functions f and g (not necessarily differentiable) can be determined from, for example, initial data $u(x, 0)$, $(\partial u/\partial t)(x, 0)$. The solution (1.2), usually referred to as *d'Alembert's solution*, then describes two distinct waves: one of which moves to the left and one to the right, both at the speed c. These waves do not interact with themselves nor with each other; this is a consequence of equation's (1.1) being *linear*, and hence solutions of the equation may be added (or *superposed*). Furthermore, the waves described by (1.2) do not change their shape as they propagate. This is easily verified if we consider one of the wave components – f say – and choose a new coordinate which is moving with this wave, $\xi = x - ct$. Then $f = f(\xi)$ and this does not change as x and t change, at fixed ξ. In other words, the shape given by $f(x)$ at $t = 0$ is exactly the same at later times but shifted to the right by an amount ct.

Before we develop some further elementary properties of waves, it is convenient first to restrict ourselves to waves which propagate in only one direction. It is clear that this is an allowable choice in solution (1.2): merely set $g \equiv 0$, for example. A more practical approach is to set up initial data on bounded (*compact*) support, and then after a finite time the two wave components f and g will move apart and no longer overlap. Since they never interact, we can now follow one of them and ignore the other. To be more specific, we may restrict the discussion to the solution of

$$u_t + cu_x = 0, \tag{1.3}$$

where we have introduced the short-hand notation for partial derivatives. The general solution of equation (1.3) is

$$u(x, t) = f(x - ct),$$

where f is an arbitrary function and, since we could redefine t as t/c, we may just as well set $c = 1$:

$$\text{if} \qquad u_t + u_x = 0 \qquad \text{then} \qquad u(x, t) = f(x - t).$$

(We may also note the connection with equation (1.1): the operator can be factorised, and either factor may be zero,

$$\left(\frac{\partial}{\partial t} \mp c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}\right)u \equiv \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0,$$

where the signs are ordered vertically.)

When wave equations are derived from some underlying physical principles (or from more general governing equations), certain simplifying assumptions are made: in extreme cases we might derive equation (1.1) or (1.3). However, if the assumptions are less extreme, we might obtain equations which retain more of the physical detail: for example, wave dispersion or dissipation, or nonlinearity. Consider, first, the equation

$$u_t + u_x + u_{xxx} = 0, \tag{1.4}$$

which is the simplest *dispersive* wave equation. To see this let us examine the form of the *harmonic* wave solution

$$u(x, t) = \mathrm{e}^{\mathrm{i}(kx - \omega t)}. \tag{1.5}$$

(We can always choose to take the real or imaginary part, or form $A\mathrm{e}^{\mathrm{i}(kx - \omega t)}$ + complex conjugate, where A is a complex constant.) Now (1.5) is a solution of equation (1.4) if

$$\omega = k - k^3; \tag{1.6}$$

this is the *dispersion relation* which determines $\omega(k)$ for given k. Here, k is the *wave number* (taken to be real so that solution (1.5) is certainly oscillatory at $t = 0$) and ω is the *frequency*. From (1.6) we see that

$$kx - \omega t = k\{x - (1 - k^2)t\},$$

and so solution (1.5), with condition (1.6), describes a wave which propagates at the velocity

$$c = \frac{\omega}{k} = 1 - k^2, \tag{1.7}$$

which is a function of k. (Note that c changes sign across $k = \pm 1$.) In other words, waves of different wave number propagate at different velocities: this is the characteristic of a *dispersive wave*. Thus a single wave profile which can be represented, let us suppose, by the sum of just two components each like solution (1.5) will change its shape as time evolves by virtue of the different velocities of the two components. However, this interpretation is virtually a repetition of the solution (1.2) of the classical wave equation. To extend the idea we need only add as many components as we desire or, for greater generality, integrate over all k to yield

$$u(x, t) = \int_{-\infty}^{\infty} A(k) e^{i\{kx - \omega(k)t\}} dk, \tag{1.8}$$

for some given $A(k)$. (Note that $A(k)$ is essentially the Fourier transform of $u(x, 0)$.) The overall effect is to produce a wave profile which changes its shape as it moves; in fact, since different components travel at different velocities the profile will necessarily spread out or *disperse*.

The velocity of an individual wave component is given by equation (1.7), and is usually termed the *phase velocity*. It is clear that equation (1.6) will admit another velocity defined by

$$c_g = \frac{d\omega}{dk} = 1 - 3k^2;$$

this is the *group velocity*, which determines the velocity of a *wave packet* (see Fig. 1.1). For many (but *not* all) realistic wave motions it turns out that

$$c_g \leqslant c$$

and, furthermore, the group velocity is the velocity of propagation of energy.

Thus far we have tacitly assumed that $\omega(k)$, the dispersion function, is real for real k. However this remains true only if we add to equation (1.4) *odd* derivatives of u with respect to x. If we choose to use *even* derivatives, taking for example

$$u_t + u_x - u_{xx} = 0, \tag{1.9}$$

then the picture is quite different. From equations (1.5) and (1.9) we obtain

$$\omega = k - \mathrm{i}k^2,$$

and so

$$u(x, t) = \exp\{-k^2 t + \mathrm{i}k(x - t)\} \tag{1.10}$$

is a solution of equation (1.9). This describes a wave which propagates at a speed of unity for all k but which also decays exponentially for any real $k\,(\neq 0)$ as $t \to +\infty$. (Note that the sign of the term u_{xx} is important.) The decay exhibited in solution (1.10) is usually called *dissipation*. Clearly we could have equations, like (1.4) or (1.9), which incorporate linear combinations of even and odd derivatives. In this case the harmonic wave solution may be both dispersive and dissipative (at least for suitable signs of the even terms).

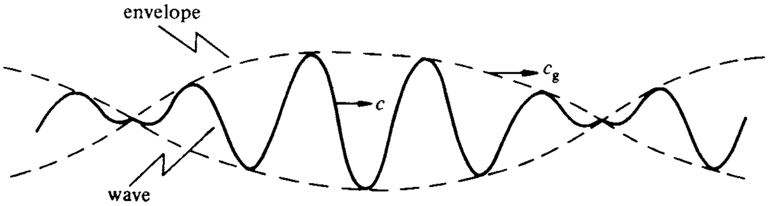
Finally, let us briefly look at one rather more involved aspect of wave motion, namely that of *nonlinearity*. Most wave equations (like (1.1) and (1.3)) are valid only for sufficiently small amplitudes. If some account is taken of the amplitude (in a 'better approximation') we might obtain the nonlinear partial differential equation

$$u_t + (1 + u)u_x = 0. \tag{1.11}$$

This equation embodies the simplest type of nonlinearity (uu_x), and comparison with equation (1.3) might suggest that it is merely a case of replacing c by $1 + u$ in the solution. It turns out (by the method of characteristics) that this is correct! From equation (1.11) we see that

$$u = \text{constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + u,$$

Fig. 1.1 A sketch of a wave packet, showing the wave and its envelope. The wave moves at the phase velocity, c, and the envelope at the group velocity, c_g.



and so the characteristic lines are $x = (1 + u)t$ + constant. Thus the general solution is

$$u(x, t) = f\{x - (1 + u)t\}, \qquad (1.12)$$

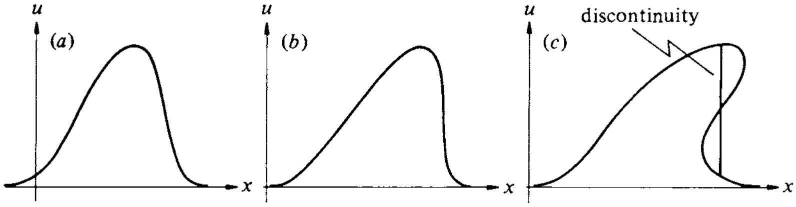
where f is an arbitrary function.

Now, given the initial wave profile, $u(x, 0) = f(x)$, it is a matter of solving equation (1.12) for u; this may be far from straightforward, even though the geometrical construction of the solution by characteristics is easy. In fact the solution of equation (1.12) (with $f > 0$, say, for some x) will generate a single-valued solution for u only for a finite time; thereafter the solution will be multi-valued (i.e. non-unique). The solution obtained by construction exhibits the non-uniqueness as a wave which has 'broken' (see Fig. 1.2). (Thus the solution must necessarily change its shape as it propagates.) This difficulty is usually overcome by the insertion of a jump (or *discontinuity*) which models a shock (again, see Fig. 1.2). Strictly, a discontinuous solution is not a proper solution of equation (1.11) but it may be allowable as a solution of the integral conservation equation from which (1.11) may have been derived.

Another complication arises with nonlinear equations: let us suppose that we have two solutions of equation (1.11), $u_1(x, t)$ and $u_2(x, t)$. We have already met the 'superposition principle' which says that, for linear equations, any linear combination of u_1 and u_2 is also a solution. However this is not true, in general, for nonlinear equations. It is easily verified that $u = u_1 + u_2$ does *not* satisfy equation (1.11). Thus solutions of nonlinear equations can not be superposed to form new solutions, although a related principle *is* available for certain nonlinear partial differential equations as we shall see.

It is clear that, by making suitable assumptions in a given physical problem, we might obtain an equation which is both nonlinear and

Fig. 1.2 The evolution of a nonlinear wave as time increases (*a*) $t = t_1$; (*b*) $t = t_2 > t_1$; (*c*) $t = t_3 > t_2$. The wave becomes vertical at one point at $t = t_2$, and thereafter the solution is triple-valued in a region. The solution can be made single-valued by the insertion of a discontinuity (and the smooth 'lobes' are then ignored).



contains dispersive or dissipative terms (or both). So, for example, we might derive

$$u_t + (1+u)u_x + u_{xxx} = 0, \tag{1.13}$$

or

$$u_t + (1+u)u_x - u_{xx} = 0. \tag{1.14}$$

The first of these is the simplest equation embodying nonlinearity and dispersion; this, or one of its elementary variants, is known as the *Korteweg–de Vries* (or *KdV*) *equation*, of which we shall say much more later. The second one, equation (1.14), with nonlinearity and dissipation, is the *Burgers equation*. In fact the general solution of equation (1.14) has been known since 1906 (Forsyth, 1906), and it turns out that there are some pointers in the method of solution which are relevant to the solution of the KdV equation. (The properties of the Burgers equation will be left to the reader to explore in the exercises.) Our main concern will be with the method of solution – and the properties of – the KdV equation, and other related 'exactly integrable' equations. However, before we embark upon a more detailed discussion, the various alternative forms of the equation should be mentioned. We can transform equation (1.13) under

$$1 + u \to \alpha u, \qquad t \to \beta t, \qquad x \to \gamma x,$$

where α, β, γ are real (non-zero) constants, to yield

$$u_t + \frac{\alpha\beta}{\gamma} u u_x + \frac{\beta}{\gamma^3} u_{xxx} = 0.$$

This is a general form of the KdV equation, and a convenient choice, which we shall often use, is

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.15}$$

Some of these transformations of variables (as used above) belong to a *continuous group* or *Lie group*. As an example, consider the transformation, G_k, of the variables x, t and u into

$$X = kx, \qquad T = k^3 t, \qquad U = k^{-2} u,$$

for real $k \neq 0$. The application of successive transformations G_k and G_l is equivalent to the single transformation G_{lk}, thereby producing the multiplication law $G_l G_k = G_{lk}$. This law is commutative since $G_k G_l = G_{kl} = G_{lk}$. Furthermore, the associative law is also satisfied because $G_k(G_l G_m) = G_k G_{lm} = G_{klm} = G_{kl} G_m = (G_k G_l) G_m$. Clearly G_1 is the identity transformation: $G_1 G_k = G_{k1} = G_k$ for all k ($\neq 0$). If we form $G_{1/k} G_k = G_1$ and $G_k G_{1/k} = G_1$, we see that $G_{1/k}$ is both the left-hand and right-hand inverse of G_k. Therefore the elements of G_k for all real $k \neq 0$ form an *infinite group*. We call k the *parameter* of this continuous group. Now let us apply the transformation G_k to the KdV equation (1.15); it becomes

$$U_T - 6UU_X + U_{XXX} = 0,$$

i.e. it is *invariant* under the continuous group of transformations, G_k. This suggests that we seek invariant properties of the solutions. In particular, we anticipate the existence of *similarity solutions* which depend only on invariant combinations of the variables (see Q1.13 and section 2.6).

We have touched on ideas associated with waves in one spatial dimension, mainly because the KdV equation (and other equations we shall meet later) take this form. Of course, waves do occur in higher dimensions; in particular the classical wave equation can be written as

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \tag{1.16}$$

where ∇^2 is the Laplace operator in the chosen coordinate system. It is clear that if we wished to examine more-complicated wave phenomena (with nonlinearity and dispersion), such as ring waves or waves crossing obliquely, then we must seek new equations. These might embody some of the character of both equations (1.15) and (1.16); in fact higher-dimensional KdV equations (and other integrable equations) do exist, but their discussion is beyond the scope of this text although one or two will be mentioned in the exercises.

1.2 The discovery of solitary waves

We have seen that the Korteweg–de Vries equation can be written down on the basis that both nonlinearity and dispersion might occur together. However, the KdV equation not only is of mathematical interest but also is of practical importance. To introduce this aspect, let us see how the solitary wave first appeared on the scientific scene. We shall then mention some of the analytical properties of this wave, and finally show that the KdV equation is indeed the relevant one for the solitary wave (and much more besides).

The solitary wave, so-called because it often occurs as this single entity and is localised, was first observed by J. Scott Russell on the Edinburgh–Glasgow canal in 1834; he called it the 'great wave of translation'. Russell reported his observations to the British Association in his 1844 'Report on Waves' in the following words:

I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

Russell also performed some laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel (see Fig. 1.3). He was able to deduce empirically that the volume of water in the wave is equal to the volume of water displaced and, further, that the speed, c, of the solitary wave is obtained from

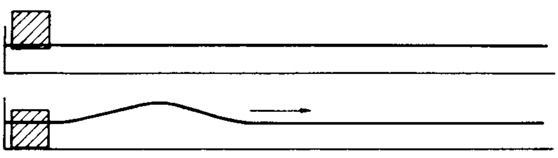
$$c^2 = g(h + a), \tag{1.17}$$

where a is the amplitude of the wave, h the undisturbed depth of water and g the acceleration of gravity (see Fig. 1.4). The solitary wave is therefore a *gravity wave*. We note immediately an important consequence of equation (1.17): higher waves travel faster. Fig. 1.3 and result (1.17) apply to waves of elevation; any attempt to generate a wave of depression results in a train of oscillatory waves, as Russell found in his own experiments.

To put Russell's formula (1.17) on a firmer footing, both Boussinesq (1871) and Lord Rayleigh (1876) assumed that a solitary wave has a length scale much greater than the depth of the water. They deduced, from the equations of motion for an inviscid incompressible fluid, Russell's formula for c. In fact they also showed that the wave profile $z = \zeta(x, t)$ is given by

$$\zeta(x, t) = a \operatorname{sech}^2 \{\beta(x - ct)\} \tag{1.18}$$

Fig. 1.3 Diagram of Scott Russell's experiment to generate a solitary wave.



where $\beta^{-2} = 4h^2(h+a)/3a$ for any $a > 0$, although the sech^2 profile is strictly only correct if $a/h \ll 1$. These authors did not, however, write down a simple equation for $\zeta(x,t)$ which admits (1.18) as a solution. This final step was completed by Korteweg & de Vries in 1895. They showed that, provided ε and σ were small, then

$$\frac{\partial \zeta}{\partial t} = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left(\frac{2}{3}\varepsilon\frac{\partial \zeta}{\partial \chi} + \zeta\frac{\partial \zeta}{\partial \chi} + \frac{1}{3}\sigma\frac{\partial^3 \zeta}{\partial \chi^3}\right), \tag{1.19}$$

where χ is a coordinate chosen to be moving (almost) with the wave. If we use the change of variables

$$\zeta = \zeta(X,t), \qquad X = \chi + \varepsilon\left(\frac{g}{h}\right)^{1/2} t$$

then equation (1.19) can be re-cast as the KdV equation

$$\zeta_t = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left(\zeta\zeta_X + \frac{1}{3}\sigma\zeta_{XXX}\right).$$

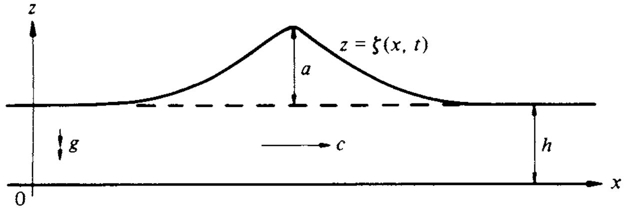
The parameter σ incorporates the surface tension, T, in the form $\sigma = \frac{1}{3}h^3 - Th/g\rho$, where ρ is the density of the liquid (and often $T \ll \frac{1}{3}g\rho h^2$); ε is an arbitrary parameter. We shall not reproduce the work of Korteweg & de Vries here, but it is instructive to see how the KdV equation arises from a set of fundamental governing equations. To this end we shall stay with water waves, but use the rather more satisfying technique of multiple-scale asymptotics.

* The governing equations of irrotational two-dimensional motion of an incompressible inviscid fluid, bounded above by a free surface and below by a rigid horizontal plane, are

$$\begin{aligned} \phi_{zz} + \delta^2\phi_{xx} &= 0; \qquad \phi_z = 0 \quad \text{on } z = 0 \\ \left.\begin{aligned} \zeta + \phi_t + \tfrac{1}{2}\alpha(\delta^{-2}\phi_z^2 + \phi_x^2) &= 0 \\ \phi_z &= \delta^2(\zeta_t + \alpha\phi_x\zeta_x) \end{aligned}\right\} &\text{ on } z = 1 + \alpha\zeta, \end{aligned} \tag{1.20}$$

where ϕ is the velocity potential. The variables used here have already

Fig. 1.4 The parameters and variables used in the description of the solitary wave.



been non-dimensionalised by the use of the undisturbed depth, h, a typical horizontal length scale, l, and the typical speed $(gh)^{1/2}$. The surface is at $z = 1 + \alpha\zeta$ and on this surface we assume that the pressure is constant (so that in this simple theory the surface tension is ignored). The parameters appearing in equations (1.20) are given by

$$\alpha = a/h, \qquad \delta = h/l,$$

where a is a measure of the wave amplitude. The two boundary conditions on $z = 1 + \alpha\zeta$ describe the constancy of pressure at the surface, and the continuity of the vertical velocity component there.

We are interested in small-amplitude long waves i.e. in the limits as $\alpha \to 0$ and $\delta \to 0$. It turns out that one choice we could make – and which leads to the KdV equation for ζ – is to set $\delta^2 = O(\alpha)$ as $\alpha \to 0$. (This seems reasonable if we note the way in which α and δ appear in equations (1.20).) Clearly, however, this is rather special and we would hope that the solitary wave is a more enduring and general phenomenon than this would suggest. It is, as the following scaling for *arbitrary* δ demonstrates. Introduce

$$\xi = \frac{\alpha^{1/2}}{\delta}(x - t), \qquad \tau = \frac{\alpha^{3/2}}{\delta}t, \qquad \Phi = \frac{\alpha^{1/2}}{\delta}\phi; \tag{1.21}$$

then equations (1.20) become

$$\begin{aligned} &\Phi_{zz} + \alpha\Phi_{\xi\xi} = 0; \qquad \Phi_z = 0 \qquad \text{on } z = 0 \\ &\left.\begin{aligned} \zeta - \Phi_\xi + \alpha\Phi_\tau + \tfrac{1}{2}(\Phi_z^2 + \alpha\Phi_\xi^2) = 0 \\ \Phi_z = \alpha(-\zeta_\xi + \alpha\zeta_\tau + \alpha\Phi_\xi\zeta_\xi) \end{aligned}\right\} \text{on } z = 1 + \alpha\zeta. \end{aligned} \tag{1.22}$$

(Note that $\xi = O(1)$, $\phi = O(1)$ and $t = O(\alpha^{-1})$ if $\delta^2 = O(\alpha)$.) The choice of variables (1.21) means that equations (1.22) hold in a frame of reference which is moving with a speed of unity to the right, and then for large times (t) if $\tau = O(1)$ as $\alpha \to 0$ (for any fixed δ). In other words, scalings (1.21) describe a particular neighbourhood of (x, t)-space where we hope the KdV equation will be valid. The appearance of a speed of unity is by virtue of the non-dimensionalisation; this corresponds to a dimensional speed of $(gh)^{1/2}$ (cf. formula (1.17) for small a). Finally, the right-ward propagation is merely for convenience: we could equally well discuss left-ward motion by introducing $\xi = \alpha^{1/2}(x + t)/\delta$.

The solution of equations (1.22), as $\alpha \to 0$, is surprisingly straightforward. To initiate the analysis we suppose that there exists a solution which takes the form

$$\Phi \sim \sum_{n=0}^{\infty} \alpha^n \Phi_n(\xi, \tau, z), \qquad \zeta \sim \sum_{n=0}^{\infty} \alpha^n \zeta_n(\xi, \tau), \qquad \text{as } \alpha \to 0,$$

for fixed ξ and τ. (Note that $z \in [0, 1 + \alpha\zeta]$ which is a bounded domain if