INTRODUCTION

For any associative ring R with unit, an abelian group $K_1(R)$ is defined as follows. For each $n > 0$, let $GL_n(R)$ denote the group of invertible $n \times n$-matrices with entries in R. Regard $GL_n(R)$ as a subgroup of $GL_{n+1}(R)$ by identifying $A \in GL_n(R)$ with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$; and set $GL(R) = \bigcup_{n=1}^{\infty} GL_n(R)$. For each n, let $E_n(R) \subseteq GL_n(R)$ be the subgroup generated by all *elementary* $n \times n$-matrices — i. e., all those which are the identity except for one nonzero off-diagonal entry — and set $E(R) = \bigcup_{n=1}^{\infty} E_n(R)$. Then by Whitehead's lemma (Theorem 1.13 below), $E(R) = [GL(R), GL(R)]$, the commutator subgroup of $GL(R)$. In particular, $E(R) \lhd GL(R)$; and the quotient group

$$K_1(R) = GL(R)/E(R)$$

is an abelian group.

One family of rings to which this applies is that of group rings. If G is any group, and if R is any commutative ring, then the group ring $R[G]$ is the free R-module with basis G, where ring multiplication is induced by the group product. In particular, group elements $g \in G$, and units $u \in R^*$, can be regarded as invertible 1×1-matrices over $R[G]$, and hence represent elements in $K_1(R[G])$. The *Whitehead group* of G is defined by setting

$$Wh(G) = K_1(\mathbb{Z}[G])/\langle \pm g \colon g \in G \rangle.$$

By construction, $K_1(R)$ (or $Wh(G)$) measures the obstruction to taking an arbitrary invertible matrix over R (or $\mathbb{Z}[G]$), and reducing it to the identity (or to some $\pm g$) via a series of *elementary operations*. Here, an elementary operation is one of the familiar matrix operations of adding a multiple of one row or column to another; and these elementary operations are very closely related to Whitehead's "elementary deformations" of finite CW complexes. This relationship leads to the definition of the *Whitehead torsion*

$$\tau(f) \in Wh(\pi_1(X))$$

of any homotopy equivalence $f: X \longrightarrow Y$ between finite CW complexes; where $\tau(f) = 1$ (i. e., the identity element) if and only if f is induced by a series of elementary deformations which transform X into Y. A homotopy equivalence f such that $\tau(f) = 1$ is called a *simple* homotopy equivalence.

Whitehead torsion plays a role, not only in studying homotopy equivalences of finite CW complexes, but also when classifying manifolds. The *s-cobordism theorem* (see Mazur [1]) says that if M and N are smooth closed n-dimensional manifolds, where $n \geq 5$, and if W is a compact (n+1)-dimensional manifold such that $\partial W = M \amalg N$, and such that the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are simple homotopy equivalences, then W is diffeomorphic to $M \times [0,1]$. In particular, M and N are diffeomorphic in this situation; and this theorem is one of the important tools for proving that two manifolds are diffeomorphic. This theorem is also one of the reasons for the importance of Whitehead groups when computing surgery obstructions.

When G is a finite group, then $K_1(\mathbb{Z}[G])$ and $Wh(G)$ are finitely generated abelian groups, whose rank was described by Bass (see the section on algorithms below, or Theorem 2.6). The main goal of this book is to develop techniques which allow a more complete description of $Wh(G)$ for finite G; and in particular which describe the subgroup

$$SK_1(\mathbb{Z}[G]) = \mathrm{Ker}\Big[K_1(\mathbb{Z}[G]) \longrightarrow K_1(\mathbb{Q}[G])\Big].$$

This is a finite subgroup (Theorem 2.5), and is in fact by a theorem of Wall (Theorem 7.4 below) isomorphic to the full torsion subgroup of $Wh(G)$. When G is abelian, then $SK_1(\mathbb{Z}[G]) = SL(\mathbb{Z}[G])/E(\mathbb{Z}[G])$, where $SL(\mathbb{Z}[G])$ denotes the group of matrices of determinant 1.

Most of the general background results have been presented here without proofs — especially when they can be referenced in standard textbooks such as Bass [2], Curtis & Reiner [1], Janusz [1], Milnor

[2], and Reiner [1]. Also, some of the longer and more technical proofs have been omitted when they are well documented in the literature, or are not needed for the central results. Proofs are included, or at least sketched, for most results which deal directly with the problem of describing Whitehead groups.

Historical survey

Whitehead groups were first defined by Whitehead [1], in order to find an algebraic analog to his "elementary deformations" of finite CW complexes, and to simple homotopy equivalences between finite CW complexes. Whitehead also showed in [1] that $Wh(G) = 1$ if $|G| \leq 4$ or if $G \cong \mathbb{Z}$; and that $Wh(C_5) \neq 1$. (Note that C_n always denotes a multilicative cyclic group of order n.)

A more systematic understanding of the structure of the groups $Wh(G)$ came only with the development of algebraic K-theory. Bass' theorem [1, Corollary 20.3], showing that the $Wh(G)$ are finitely generated and computing their rank, has already been mentioned. This made it natural to focus attention on the torsion subgroup of $Wh(G)$: shown by Higman [1] and Wall [1] to be isomorphic to $SK_1(\mathbb{Z}[G])$.

Milnor, in [1, Appendix A], noted that if the "congruence subgroup problem" could be proven, then it would follow that $SK_1(\mathbb{Z}[G]) = 1$ for all finite abelian groups G. This conjecture was shown by Bass, Milnor, and Serre [1] to be false (see Section 4c below); but their results were still sufficient to show that $SK_1(\mathbb{Z}[G])$ vanishes for many abelian groups. In particular, it was shown that $SK_1(\mathbb{Z}[G]) = 1$ if G is cyclic (Bass et al [1, Proposition 4.14]), if $G \cong C_{p^n} \times C_p$ for any prime p and any n (Lam, [1, Theorem 5.1.1]), or if $G \cong (C_2)^n$ for some n (Bass et al [1, Corollary 4.13]).

The first examples of finite groups for which $SK_1(\mathbb{Z}[G]) \neq 1$ were constructed by Alperin, Dennis, and Stein [1]. They combined earlier results from the solution to the congruence subgroup problem with theorems about generators for K_2 of finite rings, to explicitly describe $SK_1(\mathbb{Z}[G])$ when $G \cong (C_p)^n$, $n \geq 3$, and p is an odd prime. In particular, $SK_1(\mathbb{Z}[G])$ is nonvanishing for all such G. Their methods were then carried further, and used to show that for finite abelian G, $SK_1(\mathbb{Z}[G]) = 1$ if and only if either $G \cong (C_2)^n$, or each Sylow subgroup of G has the form C_{p^n} or $C_{p^n} \times C_p$.

Later results of Obayashi [1,2], Keating [1,2], and Magurn [1,2], showed that $SK_1(\mathbb{Z}[G])$ vanishes for many nonabelian metacyclic groups G, and in particular when G is any dihedral group. These were proven using various ad hoc methods, which did not give much hope for having generalizations to arbitrary finite groups. To get general results, a more systematic approach using localization sequences is needed — extending the methods of Alperin, Dennis, and Stein — and it is that approach which is the main focus of this book.

Algorithms for describing Wh(G)

If R is any commutative ring, then the usual matrix determinant induces a homomorphism

$$\det : K_1(R) = GL(R)/E(R) \longrightarrow R^*.$$

This is split surjective — split by the homomorphism $R^* \longrightarrow K_1(R)$ induced by identifying $R^* = GL_1(R)$. Hence, in this case, $K_1(R)$ factors as a product

$$K_1(R) = R^* \times SK_1(R),$$

where

$$SK_1(R) = SL(R)/E(R) \quad \text{and} \quad SL(R) = \{A \in GL(R) : \det(A) = 1\}.$$

If $R = \mathbb{Z}[G]$, then this coincides with the definition of $SK_1(\mathbb{Z}[G])$ given earlier: $\mathbb{Q}[G]$ is a product of fields, so $K_1(\mathbb{Q}[G]) \cong (\mathbb{Q}[G])^*$.

Determinants are not, in general, defined for noncommutative rings. However, in the case of the group rings $\mathbb{Z}[G]$ and $\mathbb{Q}[G]$ for finite groups G, they can be replaced by certain analogous homomorphisms: the reduced norm homomorphisms. One way to do this is to consider, for fixed G, the Wedderburn decomposition

$$\mathbb{C}[G] \cong \prod_{i=1}^{k} M_{r_i}(\mathbb{C})$$

of the complex group ring as a product of matrix rings (see Theorem 1.1). For each n, the reduced norm on $GL_n(\mathbb{Q}[G])$ is then defined to be the composite

$$nr : GL_n(\mathbb{Q}[G]) \xrightarrow{\text{incl}} GL_n(\mathbb{C}[G]) \cong \prod_{i=1}^{k} GL_{n\cdot r_i}(\mathbb{C}) \xrightarrow{\Pi \det} \prod_{i=1}^{k} \mathbb{C}^*.$$

These then factor through homomorphisms

$$nr_{\mathbb{Z}[G]} : K_1(\mathbb{Z}[G]) \longrightarrow \prod_{i=1}^{k} \mathbb{C}^* \quad \text{and} \quad nr_{\mathbb{Q}[G]} : K_1(\mathbb{Q}[G]) \longrightarrow \prod_{i=1}^{k} \mathbb{C}^*.$$

Also, $nr_{\mathbb{Q}[G]}$ is injective (Theorem 2.3), and so

$$SK_1(\mathbb{Z}[G]) = \text{Ker}\Big[K_1(\mathbb{Z}[G]) \longrightarrow K_1(\mathbb{Q}[G])\Big] \qquad \text{(by definition)}$$

$$= \text{Ker}(nr_{\mathbb{Z}[G]}). \qquad (1)$$

Note that when G is commutative, then

$$\text{Ker}(nr_{\mathbb{Z}[G]}) = \text{Ker}\Big[\det: K_1(\mathbb{Z}[G]) \longrightarrow (\mathbb{Z}[G])^*\Big];$$

so that the two definitions of $SK_1(\mathbb{Z}[G])$ coincide in this case. For more details about reduced norms (and in more generality), see Section 2a.

Reduced norm homomorphisms are also the key to computing the ranks of the finitely generated groups $K_1(\mathbb{Z}[G])$ and $Wh(G)$. Not only is $SK_1(\mathbb{Z}[G]) = \text{Ker}(nr_{\mathbb{Z}[G]})$ finite, but — once the target group has been restricted appropriately — $\text{Coker}(nr_{\mathbb{Z}[G]})$ is also finite (Theorem 2.5).

A straightforward computation using Dirichlet's unit theorem then yields the formula

$$\mathrm{rk}(K_1(\mathbb{Z}[G])) = \mathrm{rk}(\mathrm{Wh}(G))$$
$$= \#\bigl(\text{irreducible } \mathbb{R}[G]\text{-modules}\bigr) - \#\bigl(\text{irreducible } \mathbb{Q}[G]\text{-modules}\bigr).$$

Furthermore, by the theorem of Higman [1] (for commutative G) and Wall [1] (in the general case),

$$\mathrm{tors}(K_1(\mathbb{Z}[G])) = \{\pm 1\} \times G^{ab} \times SK_1(\mathbb{Z}[G])$$

(see Theorem 7.4 below). Thus, as abstract groups, at least, the structure of $K_1(\mathbb{Z}[G])$ and $\mathrm{Wh}(G)$ will be completely understood once the structure of the finite group $SK_1(\mathbb{Z}[G])$ is known.

A much more difficult problem arises if one needs to construct explicit generators for the torsion free group $\mathrm{Wh}'(G) = \mathrm{Wh}(G)/SK_1(\mathbb{Z}[G])$. One case where it is possible to get relatively good control of this is when G is a p-group, for some regular prime p (including the case $p=2$). In this case, logarithmic methods can be used to identify the p-adic completion $\hat{\mathbb{Z}}_p \otimes \mathrm{Wh}'(G)$ with a certain subgroup of $H_0(G;\hat{\mathbb{Z}}_p[G])$ (i. e., the free $\hat{\mathbb{Z}}_p$-module with basis the set of conjugacy classes in G). This is explained, and some applications are given, in Chapter 10; based on Oliver & Taylor [1, Section 4].

$SK_1(\mathbb{Z}[G])$: When studying $SK_1(\mathbb{Z}[G])$, it is convenient to first define a certain subgroup $Cl_1(\mathbb{Z}[G]) \subseteq SK_1(\mathbb{Z}[G])$. For each prime p, let $\hat{\mathbb{Z}}_p[G]$ and $\hat{\mathbb{Q}}_p[G]$ denote the p-adic completions of $\mathbb{Z}[G]$ and $\mathbb{Q}[G]$ (see Section 1b); and set $SK_1(\hat{\mathbb{Z}}_p[G]) = \mathrm{Ker}[K_1(\hat{\mathbb{Z}}_p[G]) \longrightarrow K_1(\hat{\mathbb{Q}}_p[G])]$. Then set

$$Cl_1(\mathbb{Z}[G]) = \mathrm{Ker}\Bigl[SK_1(\mathbb{Z}[G]) \xrightarrow{\ \ell\ } \bigoplus_p SK_1(\hat{\mathbb{Z}}_p[G])\Bigr].$$

The sum $\oplus_p SK_1(\hat{\mathbb{Z}}_p[G])$ is, in fact, a finite sum — $SK_1(\hat{\mathbb{Z}}_p[G]) = 1$

whenever $p \nmid |G|$ — and the localization homomorphism ℓ is onto (Theorem 3.9). Note that $Cl_1(\mathbb{Z}[G]) = SK_1(\mathbb{Z}[G])$ if G is abelian: $K_1(\hat{\mathbb{Z}}_p[G]) \cong SK_1(\hat{\mathbb{Z}}_p[G]) \times (\hat{\mathbb{Z}}_p[G])^*$ in this case, and matrices over a $\hat{\mathbb{Z}}_p$-algebra can always be diagonalized using elementary row and column operations (see Theorem 1.14(i)).

In particular, $SK_1(\mathbb{Z}[G])$ sits in an extension

$$1 \longrightarrow Cl_1(\mathbb{Z}[G]) \longrightarrow SK_1(\mathbb{Z}[G]) \xrightarrow{\ \ell\ } \bigoplus_p SK_1(\hat{\mathbb{Z}}_p[G]) \longrightarrow 1. \qquad (2)$$

The groups $SK_1(\hat{\mathbb{Z}}_p[G])$ and $Cl_1(\mathbb{Z}[G])$ are described independently, using very different methods, and it is difficult to find a way of handling them both simultaneously. In fact, one of the remaining problems is to understand the extension (2) in 2-torsion (it does have a natural splitting in odd torsion).

$SK_1(\hat{\mathbb{Z}}_p[G])$: By a theorem of Wall [1, Theorem 2.5], $SK_1(\hat{\mathbb{Z}}_p[G])$ is a p-group for any prime p and any finite group G, and $SK_1(\hat{\mathbb{Z}}_p[G]) = 1$ if the p-Sylow subgroup of G is abelian. In fact, for most "familiar" groups G, $SK_1(\hat{\mathbb{Z}}_p[G]) = 1$ for all p.

If G is a p-group, then

$$SK_1(\hat{\mathbb{Z}}_p[G]) \cong H_2(G)/H_2^{ab}(G); \qquad (3)$$

where

$$H_2^{ab}(G) = \mathrm{Im}\Big[\textstyle\sum\{H_2(K) : K \subseteq G, \ \ K \text{ abelian}\} \xrightarrow{\ \mathrm{ind}\ } H_2(G)\Big]$$

$$= \langle g \wedge h \in H_2(G) : g,h \in G, \ \ gh = hg \rangle$$

(see Section 8a). Formula (3) is shown in Theorem 8.6, and the isomorphism itself is described in Proposition 8.4.

If G is an arbitrary finite group, and if p is a fixed prime, then set $G_r = \{g \in G : p \nmid |g|\}$ (the "p-regular" elements). Consider the homology group $H_2(G;\hat{\mathbb{Z}}_p(G_r))$, where G acts on $\hat{\mathbb{Z}}_p(G_r)$ by conjugation. Let

$$\Phi : H_2(G;\hat{\mathbb{Z}}_p(G_r)) \longrightarrow H_2(G;\hat{\mathbb{Z}}_p(G_r))$$

be induced by the endomorphism $\Phi(\sum r_i g_i) = \sum r_i g_i^p$ on $\hat{\mathbb{Z}}_p(G_r)$; and let

$$H_2(G;\hat{\mathbb{Z}}_p(G_r))_\Phi = H_2(G;\hat{\mathbb{Z}}_p(G_r))/\mathrm{Im}(1-\Phi)$$

be the group of Φ-coinvariants. Then

$$SK_1(\hat{\mathbb{Z}}_p[G]) \cong H_2(G;\hat{\mathbb{Z}}_p(G_r))_\Phi/H_2^{ab}(G;\hat{\mathbb{Z}}_p(G_r))_\Phi \tag{4}$$

(see Theorem 12.10). Here, in analogy with the p-group case:

$$H_2^{ab}(G;\hat{\mathbb{Z}}_p(G_r))_\Phi = \mathrm{Im}\Big[\sum_{\substack{K \subseteq G\\ \text{abelian}}} H_2(K;\hat{\mathbb{Z}}_p(K_r)) \xrightarrow{\text{ind}} H_2(G;\hat{\mathbb{Z}}_p(G_r))_\Phi\Big]$$

$$= \langle (g\wedge h)\otimes k : g,h \in G,\ k \in G_r,\ \ g,h,k \text{ commute pairwise}\rangle.$$

The following alternative description of $SK_1(\hat{\mathbb{Z}}_p[G])$, for a non-p-group G, is often easier to use. Let $g_1,\ldots,g_k \in G$ be "$\hat{\mathbb{Q}}_p$-conjugacy" class representatives for elements of G of order prime to p — where two elements $g,h \in G$ are $\hat{\mathbb{Q}}_p$-conjugate if g is conjugate to h^{p^n} for some n. Set $Z_i = C_G(g_i)$ (the centralizer), and

$$N_i = \{x \in G : xg_ix^{-1} = g_i^{p^n},\ \text{ some } n\}.$$

Then by Theorem 12.5 below,

$$SK_1(\hat{\mathbb{Z}}_p[G]) \cong \bigoplus_{i=1}^{k} H_0\big(N_i/Z_i;\ H_2(Z_i)/H_2^{ab}(Z_i)\big)_{(p)}. \tag{5}$$

$Cl_1(\mathbb{Z}[G])$: The subgroup $Cl_1(\mathbb{Z}[G]) \subseteq SK_1(\mathbb{Z}[G])$ can be thought of as the part of $K_1(\mathbb{Z}[G])$ which is hit from behind in localization sequences. One way to study this is to consider, for any ideal $I \subseteq \mathbb{Z}[G]$ of finite index, the relative exact sequence

$$K_2(\mathbb{Z}[G]/I) \longrightarrow SK_1(\mathbb{Z}[G],I) \longrightarrow SK_1(\mathbb{Z}[G]) \longrightarrow K_1(\mathbb{Z}[G]/I)$$

of Milnor [2, Lemma 4.1 and Theorem 6.2]. After taking inverse limits over all such I, this takes the form of a new exact sequence

$$\bigoplus_p K_2^c(\hat{\mathbb{Z}}_p[G]) \longrightarrow \varprojlim_I SK_1(\mathbb{Z}[G],I) \xrightarrow{\partial} SK_1(\mathbb{Z}[G]) \xrightarrow{\ell} \bigoplus_p SK_1(\hat{\mathbb{Z}}_p[G]). \quad (6)$$

We now have another characterization of $Cl_1(\mathbb{Z}[G])$: it is the set of elements in $SK_1(\mathbb{Z}[G])$ which can be represented by matrices congruent to 1 mod I, for arbitrarily small ideals $I \subseteq \mathbb{Z}[G]$ of finite index.

The second term in (6) remains unchanged when $\mathbb{Z}[G]$ is replaced by any other $\mathbb{Z}$-order in $\mathbb{Q}[G]$. Hence, it is convenient to define

$$C(\mathbb{Q}[G]) = \varprojlim_I SK_1(\mathbb{Z}[G],I) \qquad \text{(all } I \subseteq \mathbb{Z}[G] \text{ of finite index)}$$

$$\cong \operatorname{Coker}\Big[K_2(\mathbb{Q}[G]) \longrightarrow \bigoplus_p K_2^c(\hat{\mathbb{Q}}_p[G])\Big]. \qquad \text{(Theorem 3.12)}$$

This is a finite group; and $C(-)$ is a functor on the category of finite dimensional semisimple $\mathbb{Q}$-algebras. See Section 3c for more details.

The computation of $C(\mathbb{Q}[G])$ is based on the solution to the congruence subgroup problem. In Theorem 4.13, it will be seen that for each simple summand A of $\mathbb{Q}[G]$ with center K,

$$C(A) \cong \begin{cases} 1 & \text{if for some } v\colon K \hookrightarrow \mathbb{R},\ \mathbb{R} \otimes_{vK} A \cong M_r(\mathbb{R}) \text{ (some } r) \\ \mu_K & \text{otherwise.} \end{cases} \quad (7)$$

Here, μ_K denotes the group of roots of unity in K. One convenient way to use this involves the complex representation ring $R_{\mathbb{C}}(G)$.

Fix a group G, and fix any *even* n such that $\exp(G)|n$. Then $K = \mathbb{Q}(\zeta_n)$ is a splitting field for G, where ζ_n is a primitive n-th root of unity, and we can identify the representation rings $R_{\mathbb{C}}(G) = R_K(G)$. The group $Gal(K/\mathbb{Q}) \cong (\mathbb{Z}/n)^*$ thus acts both on $R_{\mathbb{C}}(G)$ (via Galois automorphisms) and on $\mathbb{Z}/n$ (by multiplication). Regard $R_{\mathbb{R}}(G)$ (the real representation ring) as a subgroup of $R_{\mathbb{C}}(G)$ in the usual way, and set $R_{\mathbb{C}/\mathbb{R}}(G) = R_{\mathbb{C}}(G)/R_{\mathbb{R}}(G)$ for short. Then we will see in Lemma 5.9 that

$$C(\mathbb{Q}[G]) \cong \Big[R_{\mathbb{C}/\mathbb{R}}(G) \otimes \mathbb{Z}/n\Big]_{(\mathbb{Z}/n)^*} \qquad \text{(i. e., } (\mathbb{Z}/n)^*\text{-coinvariants)} \tag{8}$$
$$= R_{\mathbb{C}/\mathbb{R}}(G)/\langle [V] - a\cdot[\gamma_a(V)]:\ V \in R_{\mathbb{C}}(G),\ (a,n)=1,\ \gamma_a \in \mathrm{Gal}(K/\mathbb{Q})\rangle.$$

This description, while somewhat complicated, has the advantage of being natural in that the induced epimorphisms

$$\begin{array}{ccc} R_{\mathbb{C}}(G) & \xrightarrow{\ \tilde{\mathcal{J}}_G\ } & C(\mathbb{Q}[G]) \\ & \searrow^{\mathcal{J}_G} & \downarrow \partial \\ & & Cl_1(\mathbb{Z}[G]) \end{array}$$

commute both with maps induced by group homomorphisms and with maps induced by restriction to subgroups (Proposition 5.2). For example, one immediate consequence of this is that $Cl_1(\mathbb{Z}[G])$ is generated by induction from elementary subgroups of G (i. e., products of cyclic groups with p-groups) — since $R_{\mathbb{C}}(G)$ is generated by elementary induction by Brauer's induction theorem.

Odd torsion in $Cl_1(\mathbb{Z}[G])$ and $SK_1(\mathbb{Z}[G])$: For any finite group G, the short exact sequence (2) has a natural splitting in odd torsion, to give a direct sum decomposition

$$SK_1(\mathbb{Z}[G])[\tfrac{1}{2}] \cong Cl_1(\mathbb{Z}[G])[\tfrac{1}{2}] \oplus \bigoplus_{p\neq 2} SK_1(\hat{\mathbb{Z}}_p[G]). \tag{9}$$

Furthermore, for odd p, there is a close relationship between the groups $K_2^c(\hat{\mathbb{Z}}_p[G])$ and $H_1(G;\hat{\mathbb{Z}}_p[G]) \cong H_1(G;\mathbb{Z}[G])_{(p)}$ (where G again acts by conjugation) — close enough so that (6) can be replaced by an isomorphism

$$Cl_1(\mathbb{Z}[G])[\tfrac{1}{2}] \cong \mathrm{Coker}\Big[H_1(G;\mathbb{Z}[G]) \xrightarrow{\ \psi_G\ } C(\mathbb{Q}[G])\Big][\tfrac{1}{2}]. \tag{10}$$