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Lectures on Stochastic Analysis: Diffusion Theory

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Introduction

These notes grow out of lectures which I gave during the fall semester of 1985 at M.I.T. My purpose has been to provide a reasonably self-contained introduction to some stochastic analytic techniques which can be used in the study of certain analytic problems, and my method has been to concentrate on a particularly rich example rather than to attempt a general overview. The example which I have chosen is the study of second order partial differential operators of parabolic type. This example has the advantage that it leads very naturally to the analysis of measures on function space and the introduction of powerful probabilistic tools like martingales. At the same time, it highlights the basic virtue of probabilistic analysis: the direct role of intuition in the formulation and solution of problems.

The material which is covered has all been derived from my book [S.&V.] (Multidimensional Diffusion Processes, Grundlehren #233, Springer-Verlag, 1979) with S.R.S. Varadhan. However, the presentation here is quite different. In the first place, the emphasis there was on generality and detail; here it is on conceptual clarity. Secondly, at the time when we wrote [S.&V.], we were not aware of the ease with which the modern theory of martingales and stochastic integration can be presented. As a result, our development of that material was a kind of hybrid between the classical ideas of K. Itô and J.L. Doob and the modern theory based on the ideas of P.A. Meyer, H. Kunita, and S. Watanabe. In these notes the modern theory is presented; and the result is, I believe, not only more general but also more understandable.

In Chapter I, I give a quick review of a few of the important facts about probability measures on Polish spaces: the existence of regular conditional probability distributions and the theory of weak convergence. The chapter ends with the introduction of Wiener measure and a brief discussion of some of the basic elementary properties of Brownian motion.

Chapter II starts with an introduction to diffusion theory via the classical route of transition probability functions coming from the fundamental solution of parabolic equations. At the end of the first section, an attempt is made to bring out the analogy between diffusions and the theory of integral curves of a vector field. In this way I have tried to motivate the formulation (made precise in Chapter III) of diffusion theory in terms of martingales, and, at the same time, to indicate the central position which martingales play in stochastic analysis. The rest of Chapter II is devoted to the elements of martingale theory and the development of stochastic integration theory. (The presentation here profitted considerably from the incorporation of some ideas which I learned in the lectures given by K. Itô at the opening session of the I.M.A. in the fall of 1985.)

In Chapter III, I formulate the martingale problem and derive some of the basic facts about its solutions. The chapter ends with a proof that the martingale problem corresponding to a strictly elliptic operator with bounded continuous coefficients is well-posed. This proof turns on an elementary fact about singular integral operators, and a derivation of this fact is given in the appendix at the end of the chapter.