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978-0-521-33645-1 - Lectures on Stochastic Analysis: Diffusion Theory
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I. Stochastic Processes and Measures on Function Space:

1. Conditional Probabilities and Transition Probability Functions:

We begin by recalling the notion of conditional expectation. Namely, given a probability space $(E,\mathcal{F},P)$ and a sub-σ-algebra $\mathcal{F}'$, the conditional expectation value $E^P[X|\mathcal{F}']$ of a function $X \in L^2(P)$ is that $\mathcal{F}'$-measurable element of $L^2(P)$ such that

$$\int_A X(\xi)P(d\xi) = \int_A E^P[X|\mathcal{F}'](\xi)P(d\xi), \quad A \in \mathcal{F}'. \tag{1.1}$$

Clearly $E^P[X|\mathcal{F}']$ exists: it is nothing more or less than the projection of X onto the subspace of $L^2(P)$ consisting of $\mathcal{F}'$-measurable P-square integrable functions. Moreover, $E^P[X|\mathcal{F}'] \geq 0$ (a.s.,P) if $X \geq 0$ (a.s.,P). Hence, if X is any non-negative $\mathcal{F}$-measurable function, then one can use the monotone convergence theorem to construct a non-negative $\mathcal{F}'$-measurable $E^P[X|\mathcal{F}']$ for which (1.1) holds; and clearly, up to a P-null set, there is only one such function. In this way, one sees that for any $\mathcal{F}$-measurable X which is either non-negative or in $L^1(P)$ there exists a P-almost-surely unique $\mathcal{F}'$-measurable $E^P[X|\mathcal{F}']$satisfying (1.1). Because $X \longmapsto E^P[X|\mathcal{F}']$ is linear and preserves non-negativity, one might hope that for each $\xi \in E$ there is a $P_\xi \in \mathbf{M}_1(E)$ (the space of probability measures on $(E,\mathcal{F})$) such that $E^P[X|\mathcal{F}'](\xi) = \int X(\eta)P_\xi(d\eta)$. Unfortunately, this hope is not fulfilled in general. However, it is fulfilled when one imposes certain topological conditions on $(E,\mathcal{F})$. Our first theorem addresses this question.

(1.2) <u>Theorem</u>: Suppose that Ω is a Polish space (i.e. Ω is a topological space which admits a complete separable metrization), and that $\mathcal{A}$ is a sub σ-algebra of $\mathcal{B}_\Omega$ (the Borel field over Ω). Given $P \in \mathbf{M}_1(\Omega)$, there is an $\mathcal{A}$-measurable map $\omega \longmapsto P_\omega \in \mathbf{M}_1(\Omega)$ such that $P(A \cap B) = \int_A P_\omega(B)\, P(d\omega)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}_\Omega$. Moreover, $\omega \longmapsto P_\omega$ is uniquely determined up to a P-null set $A \in \mathcal{A}$. Finally, if $\mathcal{A}$ is countably generated, then $\omega \longmapsto P_\omega$ can be chosen so that $P_\omega(A) = \chi_A(\omega)$ for all $\omega \in \Omega$ and $A \in \mathcal{A}$.

<u>Proof</u>: Assume for the present that Ω is compact, the general case is left for later (c.f. exercise (2.2) below).

Choose $\{\varphi_n : n \geq 0\} \subseteq C(\Omega)$ to be a linearly independent set of functions whose span is dense in $C(\Omega)$, and assume that $\varphi_0 \equiv 1$. For each $n \geq 0$, let ψ_k be a bounded version of $E^P[\varphi_k | \mathcal{A}]$ and choose $\psi_0 \equiv 1$. Next, let A be the set of ω's such that there is an $n \geq 0$ and $a_0, \ldots, a_n \in \mathbb{Q}$ (the rationals) such that $\sum_{m=0}^{n} a_m \varphi_m \geq 0$ but $\sum_{m=0}^{n} a_m \psi_m(\omega) < 0$, and check that A is an $\mathcal{A}$-measurable P-null set. For $n \geq 0$ and $a_0, \ldots, a_n \in \mathbb{Q}$, define $\Lambda_\omega(\sum_{m=0}^{n} a_m \varphi_m) = E^P[\sum_{m=0}^{n} a_m \varphi_m]$ if $\omega \in A$ and $\sum_{m=0}^{n} a_m \psi_m(\omega)$ otherwise. Check that, for each $\omega \in \Omega$, Λ_ω determines a unique non-negative linear functional on $C(\Omega)$ and that $\Lambda_\omega(1) = 1$. Further, check that $\omega \longmapsto \Lambda_\omega(\varphi)$ is $\mathcal{A}$-measurable for each $\varphi \in C(\Omega)$. Finally, let P_ω be the measure on Ω associated with Λ_ω by the Riesz representation theorem and check that

$\omega \longmapsto P_\omega$ satisfies the required conditions.

The uniqueness assertion is easy. Moreover, since $P_\cdot(A) = \chi_A(\cdot)$ (a.s.,P) for each $A \in \mathcal{A}$, it is clear that, when $\mathcal{A}$ is countably generated, $\omega \longmapsto P_\omega$ can be chosen so that this equality holds for all $\omega \in \Omega$.

Q.E.D.

Referring to the set-up described in Theorem(1.2), the map $\omega \longmapsto P_\omega$ is called a <u>conditional probability distribution of P given $\mathcal{A}$</u> (abbreviated by <u>c.p.d. of P|$\mathcal{A}$</u>). If $\omega \longmapsto P_\omega$ has the additional property that $P_\omega(A) = \chi_A(\omega)$ for all $\omega \in \Omega$ and $A \in \mathcal{A}$, then $\omega \longmapsto P_\omega$ is called a <u>regular c.p.d. of P|$\mathcal{A}$</u> (abbreviated by <u>r.c.p.d. of P|$\mathcal{A}$</u>).

(1.3) <u>Remark</u>: The Polish space which will be the center of most of our attention in what follows is the space $\Omega = C([0,\infty);\mathbb{R}^N)$ of continuous paths from $[0,\infty)$ into $\mathbb{R}^N$ with the topology of uniform convergence on compact time intervals. Letting $x(t,\omega) = \omega(t)$ denote the position of $\omega \in \Omega$ at time $t \geq 0$, set $\mathcal{M}_t = \sigma(x(s): 0 \leq s \leq t)$ (the smallest σ-algebra over Ω with respect to which all the maps $\omega \longmapsto x(s,\omega)$, $0 \leq s \leq t$, are measurable). Given $P \in \mathbf{M}_1(\Omega)$, Theorem (1.2) says that for each $t \geq 0$ there is a P-essentially unique r.c.p.d. $\omega \longmapsto P_\omega^t$ of $P|\mathcal{M}_t$. Intuitively, the representation $P = \int P_\omega^t \, P(d\omega)$ can be thought of as a fibering of P according to how the path ω behaves during the initial time interval $[0,T]$. We will be mostly concerned with P's which are <u>Markov</u> in the sense that for each $t \geq 0$ and $B \in \mathcal{B}_\Omega$ which is measurable with respect to $\sigma(x(s): s \geq t)$, $\omega \longmapsto P_\omega^t(B)$ depends P-almost surely only on $x(t,\omega)$ and not on $x(s,\omega)$ for $s < t$.

2. The Weak Topology:

(2.1) Theorem: Let Ω be a Polish space and let ρ be a metric on Ω for which (Ω,ρ) is totally bounded. Suppose that Λ is a non-negative linear functional on $U(\Omega,\rho)$ (the space of ρ-uniformly continuous functions on Ω) satisfying $\Lambda(1) = 1$. Then there is a (unique) $P \in \mathbf{M}_1(\Omega)$ such that $\Lambda(\varphi) = E^P[\varphi]$ for all $\varphi \in U(\Omega,\rho)$ if and only if for all $\epsilon > 0$ there is a $K_\epsilon \subset\subset \Omega$ ("$\subset\subset$" is used to abbreviate "compact subset of") with the property that $\Lambda(\varphi) \geq 1 - \epsilon$ whenever $\varphi \in U(\Omega,\rho)$ satisfies $\varphi \geq \chi_{K_\epsilon}$.

Proof: Suppose that P exists. Choose $\{\omega_k\}$ to be a countable dense subset of Ω and for each $n \geq 1$ choose N_n so that $P(\bigcup_1^{N_n} B(\omega_k,1/n)) \geq 1 - \epsilon/2^n$, where the balls $B(\omega,r)$ are defined relative to a complete metric on Ω. Set $K_\epsilon = \bigcap_n \overline{\bigcup_1^{N_n} B(\omega_k,1/n)}$. Then $K_\epsilon \subset\subset \Omega$ and $P(K_\epsilon) \geq 1 - \epsilon$.

Next, suppose that $\Lambda(\varphi) \geq 1 - \epsilon$ whenever $\varphi \geq \chi_{K_\epsilon}$. Clearly we may assume that K_ϵ increases with decreasing ϵ. Let $\overline{\Omega}$ denote the completion of Ω with respect to ρ. Then $\varphi \in U(\Omega,\rho) \longmapsto \overline{\varphi}$, the unique extention of φ to $\overline{\Omega}$ in $C(\overline{\Omega})$, is an isometry from $U(\Omega,\rho)$ onto $C(\overline{\Omega})$. Hence, Λ induces a unique $\overline{\Lambda} \in C(\overline{\Omega})^*$ such that $\overline{\Lambda}(\overline{\varphi}) = \Lambda(\varphi)$, and so there is a $\overline{P} \in \mathbf{M}_1(\overline{\Omega})$ such that $\Lambda(\overline{\varphi}) = E^{\overline{P}}[\overline{\varphi}]$, $\varphi \in U(\Omega,\rho)$. Clearly, $\overline{P}(\Omega') = 1$ where $\Omega' = \bigcup_{\epsilon>0} K_\epsilon$, and so $P(\Gamma) = \overline{P}(\Gamma\cap\Omega')$ determines an element of $\mathbf{M}_1(\Omega)$ with the required property. The uniqueness of P is obvious.

Q.E.D.

(2.2) Exercise: Using the preceding, carry out the proof of Theorem (1.2) when Ω is not compact.

Given a Polish space Ω, the weak topology on $\mathbf{M}_1(\Omega)$ is the topology generated by sets of the form

$$\{\nu: |\nu(\varphi) - \mu(\varphi)| < \epsilon\},$$

for $\mu \in \mathbf{M}_1(\Omega)$, $\varphi \in C_b(\Omega)$, and $\epsilon > 0$. Thus, the weak topology on $\mathbf{M}_1(\Omega)$ is precisely the relative topology which $\mathbf{M}_1(\Omega)$ inherits from the weak* topology on $C_b(\Omega)^*$.

(2.3) Exercise: Let $\{\omega_k\}$ be a countable dense subset of Ω. Show that the set of convex combinations of the point masses δ_{ω_k} with non-negative rational coefficients is dense in $\mathbf{M}_1(\Omega)$. In particular, conclude that $\mathbf{M}_1(\Omega)$ is separable.

(2.4) Lemma: Given a net $\{\mu_\alpha\}$ in $\mathbf{M}_1(\Omega)$, the following are equivalent:

i) $\mu_\alpha \longrightarrow \mu$;

ii) ρ is a metric on Ω and $\mu_\alpha(\varphi) \longrightarrow \mu(\varphi)$ for every $\varphi \in U(\Omega,\rho)$;

iii) $\overline{\lim_\alpha}\, \mu_\alpha(F) \leq \mu(F)$ for every closed F in Ω;

iv) $\underline{\lim_\alpha}\, \mu_\alpha(G) \geq \mu(G)$ for every open G in Ω;

v) $\lim_\alpha \mu(\Gamma) = \mu(\Gamma)$ for every $\Gamma \in \mathscr{B}_\Omega$ with $\mu(\partial\Gamma) = 0$.

Proof: Obviously i) implies ii) and iii) implies iv) implies v). To prove that ii) implies iii), set

$$\varphi_\epsilon(\omega) = \rho(\omega,(F^{(\epsilon)})^c)/[\rho(\omega,F) + \rho(\omega,(F^{(\epsilon)})^c)],$$

where $F^{(\epsilon)}$ is the set of ω's whose ρ-distance from F is less than ϵ. Then, $\varphi_\epsilon \in U(\Omega,\rho)$, $\chi_F \leq \varphi_\epsilon \leq \chi_{F^{(\epsilon)}}$, and so:

$$\overline{\lim_{\alpha}}\ \mu_\alpha(F) \le \lim_{\alpha}\ \mu_\alpha(\varphi_\epsilon) = \mu(\varphi_\epsilon) \le \mu(F^{(\epsilon)}).$$

After letting $\epsilon \longrightarrow 0$, one sees that iii) holds.

Finally, assume v) and let $\varphi \in C_b(\Omega)$ be given. Noting that $\mu(a < \varphi < b) = \mu(a \le \varphi \le b)$ for all but at most a countable number of a's and b's, choose for a given $\epsilon > 0$ a finite collection $a_0 <\ldots< a_N$ so that $a_0 < \varphi < a_N$, $a_n - a_{n-1} < \epsilon$, and $\mu(a_{n-1} < \varphi < a_n) = \mu(a_{n-1} \le \varphi \le a_n)$ for $1 \le n \le N$. Then:

$$|\mu_\alpha(\varphi) - \mu(\varphi)| \le$$
$$2\epsilon + 2\|\varphi\|_{C_b(\Omega)} \sum_1^N |\mu_\alpha(a_{n-1} < \varphi \le a_n) - \mu(a_{n-1} < \varphi \le a_n)|,$$

and so, by v), $\overline{\lim_{\alpha}}\ |\mu_\alpha(\varphi) - \mu(\varphi)| \le 2\epsilon$. Q.E.D.

(2.5) Remark: $\mathbf{M}_1(\Omega)$ admits a metric. Indeed, let ρ be a metric on Ω with the property that (Ω,ρ) is totally bounded. Then, since $U(\Omega,\rho)$ is isometric to $C(\overline{\Omega})$, there is a countable dense subset $\{\varphi_n\}$ of $U(\Omega,\rho)$. Define

$$R(\mu,\nu) = \sum_1^\infty |\mu(\varphi_n) - \nu(\varphi_n)|/2^n(1 + \|\varphi_n\|_{C_b(\Omega)}).$$

Clearly R is a metric for $\mathbf{M}_1(\Omega)$, and so (in view of (2.3)) we now see that $\mathbf{M}_1(\Omega)$ is a separable metric space. Actually, with a little more effort, one can show that $\mathbf{M}_1(\Omega)$ is itself a Polish space. The easiest way to see this is to show that $\mathbf{M}_1(\Omega)$ can be embedded in $\mathbf{M}_1(\overline{\Omega})$ as a G_δ. Since $\mathbf{M}_1(\overline{\Omega})$ is compact, and therefore Polish, it follows that $\mathbf{M}_1(\Omega)$ is also Polish. In any case, we now know that convergence and sequential convergence are equivalent in $\mathbf{M}_1(\Omega)$.

(2.6) Theorem(Prokhorov & Varadarajan): A set $\Gamma \subseteq \mathbf{M}_1(\Omega)$ is relatively compact if and only if for each $\epsilon > 0$ there is a $K_\epsilon \subset\subset \Omega$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$ for every $\mu \in \Gamma$.

Proof: First suppose that $\Gamma \subset\subset \mathbf{M}_1(\Omega)$. Given $\epsilon > 0$ and $n \geq 1$, choose for each $\mu \in \Gamma$ a $K_n(\mu) \subset\subset \Omega$ so that $\mu(K_n(\mu)) > 1 - \epsilon/2^n$ and set $G_n(\mu) = \{\nu\colon \nu(K_n(\mu)^{(1/n)}) > 1 - \epsilon/2^n\}$, where distances are taken with respect to a complete metric on Ω. Next, choose $\mu_{n,1},\ldots,\mu_{n,N_n} \in \Gamma$ so that $\Gamma \subseteq \bigcup_{k=1}^{N_n} G_n(\mu_{n,k})$, and set $K = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=1}^{N_n} K_n(\mu_{n,k})^{(1/n)}}$. Clearly $K \subset\subset \Omega$ and $\mu(K) \geq 1 - \epsilon$ for every $\mu \in \Gamma$.

To prove the opposite implication, think of $\mathbf{M}_1(\Omega)$ as a subset of the unit ball in $C_b(\Omega)^*$. Since the unit ball in $C_b(\Omega)^*$ is compact in the weak* topology, it suffices for us to check that every weak* limit Λ of μ's from Γ comes from an element of $\mathbf{M}_1(\Omega)$. But $\Lambda(\varphi) \geq 1 - \epsilon$ for all $\varphi \in C_b(\Omega)$ satisfying $\varphi \geq \chi_{K_\epsilon}$, and so Theorem(2.1) applies to Λ.
Q.E.D.

(2.7) Example: Let $\Omega = C([0,\infty);E)$, where (E,ρ) is a Polish space and we give Ω the topology of uniform convergence on finite intervals. Then, $\Gamma \subseteq \mathbf{M}_1(\Omega)$ is relatively compact if and only if for each $T > 0$ and $\epsilon > 0$ there exist $K \subset\subset E$ and $\delta\colon(0,\infty)\longrightarrow(0,\infty)$, satisfying $\lim_{\tau\downarrow 0} \delta(\tau) = 0$, such that:

$$\sup_{P\in\Gamma} P(\{\omega\colon x(t,\omega) \in K,\ t \in [0,T],\ \text{and}\ \rho(x(t,\omega),x(s,\omega)) \leq \delta(|t-s|),\ s,t \in [0,T]\}) \geq 1 - \epsilon.$$

In particular, if ρ-bounded subsets of E are relatively compact, then it suffices that:

$$\overline{\lim_{R\to\infty}} \sup_{P\in\Gamma} P(\{\omega: \rho(x,x(0,\omega)) \le R \text{ and } \rho(x(t,\omega),x(s,\omega)) \le \delta(|t-s|),\ s,t \in [0,T]\}) \ge 1-\varepsilon$$

for some reference point $x \in E$.

The following basic real-variable result was discovered by Garsia, Rademich, and Rumsey.

(2.8) Lemma (Garsia et al.): Let p and Ψ be strictly increasing continuous functions on $(0,\infty)$ satisfying $p(0) = \Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. For given $T > 0$ and $\varphi \in C([0,T]:\mathbb{R}^N)$, suppose that:

$$\int_0^T\int_0^T \Psi(|\varphi(t) - \varphi(s)|/p(|t-s|))dsdt \le B < \infty.$$

Then, for all $0 \le s \le t \le T$:

$$|\varphi(t) - \varphi(s)| \le 8\int_0^{t-s} \Psi^{-1}(4B/u^2)p(du).$$

Proof: Define

$$I(t) = \int_0^T \Psi(|\varphi(t) - \varphi(s)|/p(|t-s|))ds,\ t \in [0,T].$$

Since $\int_0^T I(t)dt \le B$, there is a $t_0 \in (0,T)$ such that $I(t_0) \le B/T$. Next, choose $t_0 > d_0 > t_1 > \ldots > t_n > d_n > \ldots$ as follows. Given t_{n-1}, define d_{n-1} by $p(d_{n-1}) = 1/2p(t_{n-1})$ and choose $t_n \in (0,d_{n-1})$ so that $I(t_n) \le 2B/d_{n-1}$ and

$$\Psi(|\varphi(t_n) - \varphi(t_{n-1})|/p(|t_n-t_{n-1}|)) \le 2I(t_{n-1})/d_{n-1}.$$

Such a t_n exists because each of the specified conditions can fail on a set of at most measure $d_{n-1}/2$.

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Clearly:

$$2p(d_{n+1}) = p(t_{n+1}) \lesssim p(d_n).$$

Thus, $t_n \downarrow 0$ as $n \longrightarrow \infty$. Also, $p(t_n - t_{n+1}) \lesssim p(t_n) = 2p(d_n) = 4(p(d_n) - 1/2p(d_n)) \lesssim 4(p(d_n) - p(d_{n+1}))$. Hence, with $d_{-1} \equiv T$:

$$\begin{aligned}|\varphi(t_{n+1}) - \varphi(t_n)| &\lesssim \Psi^{-1}(2I(t_n)/d_n)p(t_n - t_{n+1}) \\ &\lesssim \Psi^{-1}(4B/d_{n-1}d_n)(p(d_n) - p(d_{n+1})) \\ &\lesssim 4\int_{d_{n+1}}^{d_n} \Psi^{-1}(4B/u^2)p(du),\end{aligned}$$

and so $|\varphi(t_0) - \varphi(0)| \lesssim 4\int_0^T \Psi^{-1}(4B/u^2)p(du)$. By the same argument going in the opposite time direction, $|\varphi(T) - \varphi(t_0)| \lesssim 4\int_0^T \Psi^{-1}(4B/u^2)p(du)$. Hence, we now have:

$$|\varphi(T) - \varphi(0)| \lesssim 8\int_0^T \Psi^{-1}(4B/u^2)p(du). \tag{2.9}$$

To complete the proof, let $0 \lesssim \sigma \lesssim \tau \lesssim T$ be given and apply (2.9) to $\bar{\varphi}(t) = \varphi(\sigma + (\tau-\sigma)t/T)$ and $\bar{p}(t) = p((\tau-\sigma)t/T)$. Since

$$\begin{aligned}&\int_0^T\int_0^T \Psi(|\bar{\varphi}(t) - \bar{\varphi}(s)|/\bar{p}(|t - s|))dsdt = \\ &(T/(\tau-\sigma))^2\int_\sigma^\tau\int_\sigma^\tau \Psi(|\varphi(t) - \varphi(s)|/p(|t - s|))dsdt \\ &\lesssim (T/(\tau-\sigma))^2 B \equiv \bar{B},\end{aligned}$$

we conclude that:

$$\begin{aligned}|\varphi(\tau) - \varphi(\sigma)| &\lesssim 8\int_0^T \Psi^{-1}(4\bar{B}/u^2)\bar{p}(du) \\ &= 8\int_0^{\tau-\sigma} \Psi^{-1}(4B/u^2)p(du).\end{aligned}$$

Q.E.D.

(2.10)<u>Exercise</u>: Generalize the preceding as follows.

Let $(L,\|\cdot\|)$ be a normed linear space, $r > 0$, $d \in Z^+$, and $\varphi:\mathbb{R}^d \longrightarrow L$ a weakly continous map. Set $B(r) = \{x\in\mathbb{R}^d: |x| < r\}$ and suppose that

$$\int_{B(r)}\int_{B(r)} \Psi(\|\varphi(y) - \varphi(x)\|/p(|y-x|))dxdy \le B < \infty.$$

Show that

$$\|\varphi(y) - \varphi(x)\| \le 8\int_0^{|y-x|} \Psi^{-1}(4^{d+1}B/\gamma u^{2d})p(du)$$

where

$$\gamma = \gamma_d \equiv \inf\{|(x + B(r))\cap B(1)|/r^d: |x| \le 1 \text{ and } r \le 2\}.$$

A proof can be made by mimicking the argument used to prove Lemma(2.9) (cf. 2.4.1 on p. 60 of [S.&V.]).

(2.12)<u>Theorem</u> (Kolmogorov's Criterion): Let $(\Omega,\mathcal{F},P)$ be a probability space and ξ a measurable map of $\mathbb{R}^d \times\Omega$ into the normed linear space $(L,\|\cdot\|)$. Assume that $x \in R^d \longmapsto \xi(x,\omega)$ is weakly continuous for each $\omega \in \Omega$ and that for some $q \in [1,\infty)$, $r > 0$, $\alpha > 0$, and $A < \infty$:

$$E^P[\|\xi(y) - \xi(x)\|^q] \le A|y - x|^{d+\alpha},\quad x,y \in B(r). \tag{2.13}$$

Then, for all $\lambda > 0$,

$$P(\sup_{x,y\in B(r)} \|\xi(y) - \xi(x)\|/|y - x|^\beta \ge \lambda) \le AB/\lambda^q \tag{2.14}$$

where $\beta = \alpha/2q$ and $B < \infty$ depends only on d, q, r, and α.

<u>Proof</u>: Let $\rho = 2d + \alpha/2$. Then:

$$\int_{B(r)}\int_{B(r)} E^P[(\|\xi(y) - \xi(x)\|/|y - x|^{\rho/q})^q)]dxdy \le AB'$$

where

$$B' \equiv \int_{B(r)}\int_{B(r)} |y - x|^{-d+\alpha/2}dxdy.$$

Next, set