# Algebraic homotopy

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*Für Barbara*
*und für unsere Kinder Charis und Sarah*

# CONTENTS

# PREFACE

This book gives a new general outlook on homotopy theory: fundamental ideas of homotopy theory are developed in the presence of a few axioms so that they are available in a broad variety of contexts. Many examples and applications in topology and algebra are discussed; we consider the homotopy theory of topological spaces, the algebraic homotopy theory of chain algebras, and rational homotopy theory.

The axiomatic approach saves a lot of work in the various fields of application and offers a new way of organizing a course of modern homotopy theory. A fruitful interplay takes place among the various applications.

This book is also a research monograph on homotopy classification problems. The main new result and our principal objective is the '**tower of categories**' which approximates the homotopy category of complexes. Such towers turn out to be a useful new tool for homotopy classification problems; they complement the well-known spectral sequences. The theory on complexes is a continuation of J.H.C. Whitehead's combinatorial homotopy. In fact, some of Whitehead's results can be derived readily from the properties of the towers.

In a later chapter (Chapter IX) we describe the simplest examples of towers of categories from which nevertheless fundamental results of homotopy theory can be immediately deduced.

Most of the material in the book does not appear in any textbook on algebraic topology and homotopy theory.

As prerequisites the reader should be familiar with elementary topology and the language of categories. The book can also be used by readers who have only a little knowledge of topology and homotopy theory, for example when they want to apply the methods of homotopical algebra in an algebraic context. The book covers the elementary homotopy theory in an abstract way.

The nine chapters which comprise the book are subdivided into several sections, §0, §1, §1a, §1b, §2, etc. Definitions, propositions, remarks, etc., are consecutively numberd in each section, each number being preceded by the section number, for example (1.5) or (1a.5). A reference such as (II. 5.6) points to (5.6) in Chapter II, while (5.6) points to (5.6) in the chapter at hand. References to the bibliography are given by the author's name, e.g. J.H.C. Whitehead (1950).

I lectured on the material presented in this book in Bonn (1982), Lille (1982), Berlin (1985), Zürich (1986) and on several conferences. There are further applications of the results which cannot be described in a book of this size. In particular, we obtained an algebraic classification of $(n-1)$-connected $(n+3)$-dimensional polyhedra for $n \geqq 1$. The invariants are computable; for example, there exist exactly 4732 simply connected homotopy types which have the homology groups $(n \geqq 4)$

$$H_i(X) = \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z} & i = n \\ \mathbb{Z}/8 \oplus \mathbb{Z} & i = n+1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z} & i = n+2 \\ \mathbb{Z} & i = n+3 \\ 0 & \text{otherwise.} \end{cases}$$

Further details will appear elsewhere. However, the basic machinery for these results is developed in this book.

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H.J. Baues
Zürich, im Mai 1986

# INTRODUCTION

In his lecture at the international congress of mathematicians (1950) J.H.C. Whitehead outlined the idea of algebraic homotopy as follows:

> In homotopy theory, spaces are classified in terms of homotopy classes of maps, rather than individual maps of one space in another. Thus, using the word category in the sense of S. Eilenberg and Saunders Mac Lane, a homotopy category of spaces is one in which the objects are topological spaces and the 'mappings' are not individual maps but homotopy classes of ordinary maps. The equivalences are the classes with two-sided inverses, and two spaces are of the same homotopy type if and only if they are related by such an equivalence. The ultimate object of **algebraic homotopy** is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry.

This goal of algebraic homotopy in particular includes the following basic *homotopy classification problems:*

> Classify the homotopy types of polyhedra $X, Y\ldots$, by algebraic data. Compute the set of homotopy classes of maps, $[X, Y]$, in terms of the classifying data for $X$ and $Y$. Moreover, compute the group of homotopy equivalences, $\mathrm{Aut}(X)$.

There is no restriction on the algebraic theory which might solve these problems, except the restriction of 'effective calculability'. Indeed, algebraic homotopy is asking for a theory which, a priori, is not known and which is not uniquely determined by the problem. Moreover, it is not clear whether there is a suitable purely algebraic theory for the problem better than the simplicial approach of Kan. For example, in spite of enormous efforts in the last four decades, there is still no successful computation of the *homotopy groups of spheres*

$$\pi_m S^n = [S^m, S^n],$$

which turned out to have a very rich structure. This example shows that the difficulties for a solution of the homotopy classification problems increase rapidly when, for the spaces involved, the

$$\textit{range} = (\text{dimension}) - (\text{degree of connectedness})$$

grows. On the other hand by a classical result of Serre, the *rational homotopy groups of spheres*

$$\pi_m(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & m = n > 0 \\ \mathbb{Q}, & m = 2n-1,\ n \text{ even} \\ 0, & \text{otherwise,} \end{cases}$$

are indeed simple objects. These remarks indicate two suitable restrictions for the homotopy classification problem: consider the problem in a small range, or consider the problem for rational spaces.

Quillen (1969) showed that a *differential Lie algebra* is an algebraic equivalent of the homotopy type of a simply connected rational space. Sullivan (1977) obtained the 'dual' result using *commutative cochain algebras* and the de Rham functor.

On the other hand, it is surprising how little is known on homotopy types of finite polyhedra. J.H.C. Whitehead (1949) showed that the cellular *chain complex* of the universal covering is an algebraic equivalent for a 3-dimensional polyhedron. Moreover, he classified simply connected 4-dimensional polyhedra by his '*certain exact sequence*'.

Using towers of categories we obtain in this book new proofs and new insights for these results of Quillen, Sullivan, and Whitehead respectively.

Algebraic models of homotopy types are often obtained by functors which carry polyhedra to algebraic objects like chain complexes, chain algebras, commutative cochain algebras, and chain Lie algebras. The categories defined by these objects yield homotopy categories in which the 'mappings' are not individual maps but homotopy classes of maps. There are actually many more algebraic homotopy categories, some of which not related to spaces at all. In each of them one has homotopy classification problems as above. It turns out that there is a striking similarity of properties of such homotopy categories (compare, for example, Chapter IX). This fact and the large number of homotopy categories make it necessary to develop a theory based on axioms which are in force in most of the homotopy categories.

The idea of axiomatizing homotopy is used implicitly by Eckmann–Hilton in studying the phenomena of duality in homotopy theory. Hilton (1965, p. 168) actually draws up a program by mentioning:

> Finally we remark that one would try to define the notions of cone, suspension, loop space, etc. for the category C and thus *place the duality on a strict logical basis*. It would seem therefore that we should consider an abstract system formalizing the category of spaces, its homotopy category and the homotopy functors connecting them.

To carry out this program is a further objective of this book. We develop homotopy theory abstractly in the presence of only four axioms on cofibrations and weak equivalences. These axioms are substantially weaker than those of Quillen. Many applications of the abstract theory and numerous examples in topology and algebra are described.

There is a wide variety of contexts where the techniques of homotopy theory are useful. Therefore, the unification due to the abstract development of the theory possesses major advantages: *one proof replaces many*; in addition, an interplay takes place among the various applications. This is fruitful for many topological and algebraic contexts. We derive from the axioms a theory which in topology can be compared with combinatorial homotopy theory in the sense of J.H.C. Whitehead.

Hence a few axioms on cofibrations and weak equivalences in a category imply a rich homotopy theory in this category. Moreover, such theories can be compared by use of functors which carry weak equivalences to weak equivalences. This leads to a wider understanding of homotopy theory and offers methods for the solution of the homotopy classification problems.

# LIST OF SYMBOLS

Numbers in brackets are page numbers.
□ indicates end of proof.
|| indicates end of definition or end of remark.

## (i) Notation for categories

Boldface letters like **C**, **F**, **K**, **A**, **B**..., denote categories

**Nat**, category of natural systems (246)
$H^n(\mathbf{C}, D)$, $\mathrm{Lim}^n$, cohomology (246, 249)
$\lambda: \mathbf{A} \rightarrowtail \mathbf{B}$, faithful functor
$f: A \rightarrowtail B$, injective function of sets
$\lambda: \mathbf{A} \twoheadrightarrow \mathbf{B}$, full functor
$f: A \twoheadrightarrow B$, surjective function of sets
$\lambda: \mathbf{A} \xrightarrow{\sim} \mathbf{B}$, equivalence of categories
$\ker(f) = \mathrm{kernel}(f) = f^{-1}(0) \subset A$
$\mathrm{im}(f) = \mathrm{image}(f) = f(A) \subset B$

## (ii) Topological and algebraic notation

**Top**, **Top***, $\mathbf{Top}_0^*$, $\mathbf{Top}_1^*$, **Top** (2, 31, 68, 445, 32)
$\mathbb{Q} \subset \mathbb{R}$ (37, 2)
$S^{n-1} \subset D^n$, $S_n$ (5, 159, 417)
$\dot{\cup}$ (36)
$SX, |SX|$ (36, 37, 104)
**CW-spaces**, **CW-spaces** $(R)$ (37, 436)
$X_R, X^+$ (37, 39)
$[\pi, \pi] \subset \pi$, $(a, b)$, $a^b$ (38, 313)
$\mathbf{Chain}_R$, $\mathbf{Chain}_R^+$ (40)
$|X|$, $V^n = V_{-n}$, $sV, HV$ (39, 40)
$V \oplus V'$ (40)
$V \otimes W$, $A \otimes B$, $A \otimes_B Y$ (40, 58, 59, 66)
$SC_*(X)$ (44)
**DA**, $\mathbf{DA}_*$, **DA**(flat), **DA**(free) (46, 47)
**CDA**, $\mathbf{CDA}_*$, $\mathbf{CDA}_*^0$ (59)
**DL**, $\mathbf{DL}_1$ (76, 79)
$\tilde{A}, Q(A)$ (45)
$[L, L], Q(L)$ (75)
$T(V)$, $T(V, dV)$ (45, 50)
$\Lambda(V)$, $\Lambda(V, dV)$ (59, 61)
$L(V)$, $L(V, dV)$ (76, 78)
$V^{\otimes n}$ (45)
$\mathrm{Mon}(E)$ (46)
$A \coprod B$, $L \coprod L'$ (46, 76)
$\Omega X, SC_*\Omega X$ (57)
$x \wedge y$ (59)
$\exp(\theta) = e^\theta$ (70, 80)
$\pi_\psi^n(A, B)$ (72)
$A_{DR}^*(M)$, $A_R(X)$, $A_Q$ (70, 418, 428)
$\Gamma_q G$ (73)
$fNQ, fMQ$ (74)
$(fn\mathbb{Z})_D$, $(ffn\mathbb{Z})_D$ (428)
$(fnQ)_D$, $(ffnQ)_D$ (428)
$U(L)$ (75)
$[x, y]$, $[\alpha, \beta]$ (75, 157)
$\pi_n(X) = \pi_n^{S^0}(X)$ (117)
$\xi^\alpha, x^\alpha$ (117, 123)
$\alpha \cup \beta$, $f \cup g$ (163, 250)
$\bigcup_{n,m}$ (163)
$w = w_{A,B}$ (163)
deg (167)
$\mathbb{C}P_n$ (167)
$BSO(n)$ (168)
$C_nX$, $\hat{C}_nX$, $\hat{C}_*(X, D)$ (199, 208, 308)
$\mathbb{Z}[\pi]$, $\mathbb{Z}[M]$, $\varphi_\#$ (206, 324)
$\mathbf{Mod}_{\mathbb{Z}}^{\wedge}$, $\mathrm{Hom}_\varphi(M, N)$, $\mathbf{Mod}_\pi$ (206, 405)
$\mathbf{Chain}_{\mathbb{Z}}^{\wedge}$ (206)
$\hat{H}_*(C, \Gamma)$, $\hat{H}^*(C, \Gamma)$ (207)
$\hat{H}_*(X, D; \Gamma)$, $\hat{H}^*(X, D; \Gamma)$ (207, 38)
$\hat{X} \to X$ (207)
$K(A, n)$, $L(A, n)$ (213, 214)
$\Lambda(V, n)$, $L(V, n)$ (416, 419)
$M(A, n)$ (267)
$\mathrm{Map}(X, E)_B^Y$ (116)
**Set** (233)
**Gr**, **Ab** (236)
$\Gamma, \Gamma(A)$ (268, 447)
$\dim(A - D) = \dim(A, D)$ (278)
$\mathrm{hodim}(B|D)$ (296)
$\pi_n(X|D)$ (298)
$\mathbf{M}^n, \mathbf{A}_n^k$ (267, 286)
$\mathbf{CW}_0^D, \mathbf{CW}_D^1$ (307, 404)
$\rho(X, D)$ (308)
$\langle Z_1 \rangle$, $A * B$, $A^{*B}$ (309, 313)
$\mathbf{H}(G)$, $\mathbf{H}_1(G)$, $\mathbf{H}_2(G)$ (311, 324, 322)
$\mathbf{H}_{n+1}, \mathbf{H}_{n+1}^c, \mathbf{rH}_{n+1}^c$ (338, 440)
$\Gamma_n(U, D), \tilde{\Gamma}_2(X, D)$ (227, 317)
**n-types** (364)
$\mathbf{t}_n$ (367)
$(\mathscr{X}_0^1)^D$, $(\mathscr{X}_k^r)^D$, $(\mathscr{E}_k^r)^D$ (372, 397, 399)
$\mathbf{T}_n$ (406)
$\mathbf{K}_D^2$, $\mathbf{K}_D^n$, $\mathbf{k}_D^{n+1}$ (405, 415)
$\mathbf{K}_n^D, \mathbf{k}_{n+1}^D$ (426)
$\mathbf{Co}_B$ (420)
$\mathrm{Tor}_n^X(R, M)$ (437)

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## (iii) List of 'dual' notations

| *Cofibration category* $\mathbf{C}$ | | *Fibration category* $\mathbf{F}$ | |
|---|---|---|---|
| $Y \xrightarrow{\sim} X \in we$ | (5) | $X \xrightarrow{\sim} Y \in we$ | (10) |
| $Y \rightarrowtail X \in \mathrm{cof}$ | (6) | $X \twoheadrightarrow Y \in \mathrm{fib}$ | (11) |
| $(f: B \to A) = (A, B) \in \mathbf{Pair}\,(\mathbf{C})$ | (85) | $(f: A \to B) = (A \mid B) \in \mathbf{Pair}\,(\mathbf{F})$ | (153) |
| $\simeq$ push | (8) | $\simeq$ pull | |
| $A \bigcup_B Y \xrightarrow{(\alpha,\beta)} U$ | (6, 83) | $U \xrightarrow{(\alpha,\beta)} A \times_B Y$ | (11) |
| $\alpha \cup \beta$ | (84) | $\alpha \times \beta$ | |
| $\varnothing = *$ | (7) | $e = *$ | (11) |
| $A + Y = A \vee Y = A \bigcup_\varnothing Y$ | (8) | $A \times Y = A \times_e Y$ | |
| $IX = I \times X$ | (2, 18) | $PX = X^I$ | (3, 28, 152) |
| $I_Y X$ | (8, 20) | $P_Y X$ | |
| $X \cup I_B A \cup X$ | (97) | $X \times P_B A \times X$ | |
| $\Sigma_Y^0 X, \Sigma_Y X, \Sigma_Y^n X$ | (107, 115, 137) | $\Omega_Y^0 X, \Omega_Y X, \Omega_Y^n X$ | (110) |
| $\Sigma \cdot f, \Sigma f$ | (107, 177, 133) | $\Omega \cdot f, \Omega f$ | |
| $0_X = 0: X \to *$ | (115) | $0_X = 0: * \to X$ | (152) |
| $CX$ | (119) | $WX$ | (152) |
| $\Sigma X, \Sigma^n X, \Sigma_B^n$ | (20, 116, 135) | $\Omega X, \Omega^n X, \Omega_B^n$ | (152) |
| $Z_f$ | (8, 23) | $W_f$ | (28) |
| $C_f = CA \bigcup_f B$ | (125) | $P_f = B \times_f WA$ | (154) |
| $f_0 \simeq f_1$ | (3, 20) | $f_0 \simeq f_1$ | (3, 28) |
| $\alpha \simeq \beta$ rel $X$, under $X$ | (8, 92) | $\alpha \simeq \beta$ over $X$ | (12) |
| $\mathrm{Hom}\,(X, U)^u$ | (92) | $\mathrm{Hom}\,(U, X)_u$ | |
| $[X, U]^Y = [X, U]^u$ | (92) | $[X, U]_Y = [X, U]_u$ | (110) |
| $[X, U]^* = [X, U]$ | (3, 92, 116) | $[X, U]_* = [X, U]$ | |
| $g_*, f^*$ | (94, 118, 137) | $g^*, f_*$ | |
| $\pi_n^A(U) = [\Sigma^n A, U]$ | (117, 137) | $\pi_B^n(U) = [U, \Omega^n B]$ | (153) |
| $\pi_{n+1}^A(U, V)$ | (120) | $\pi_B^{n+1}(U \mid V)$ | (153) |
| $\pi_n^X(A \vee B)_2$ | (142) | $\pi_X^n(A \times B)_2$ | (153) |
| $E_g, E, (\pi_g, 1)$ | (143) | $L_g, L, (\pi_g, 1)$ | (154) |
| $\nabla f, \nabla, \nabla_F$ | (145, 146, 206, 314) | $\nabla f, \nabla, \nabla_F$ | |
| $\nabla(v, f), \nabla^{n+1}(u, f)$ | (151) | $\nabla(u, f), \nabla^{n+1}(u, f)$ | |
| $w^+$ | (126) | $w^+$ | |
| $\mathrm{d}(u_0, H, u_1), \mathrm{d}(u_0, u_1)$ | (127) | $\mathrm{d}(u_0, H, u_1), \mathrm{d}(u_0, u_1)$ | |
| $\Sigma(w, f)$ | (127) | $\Omega(w, f)$ | |
| $B/A$ cofiber | (128) | fiber | |
| $\Sigma: [A, X]_0 \to [\Sigma A, \Sigma X]_0$ | (133) | $\Omega: [X, A]_0 \to [\Omega X, \Omega A]_0$ | |
| $\alpha_L$ | (111, 118, 141) | $\alpha_L$ | |
| $H_Y(x, y) = [I_Y X, U]^{(x,y)}$ | (105) | $H_Y(x, y) = [U, P_Y X]_{(x,y)}$ | |
| $O, -G, H + G$ | (93, 105) | $O, -G, H + G$ | |
| $G^\#$ | (106, 138) | $G^\#$ | |
| $[\Sigma_Y X, U]^u = \pi_1(U^{X \mid Y}, u)$ | (107) | $[U, \Omega_Y X]_u = \pi_1((X \mid Y)^U, u)$ | |
| $U^{X \mid Y}$ | (108, 137) | $(X \mid Y)^U$ | |
| $u^+(H) = u + H = H^\#(u)$ | (109) | $u^+(H) = u + H = H^\#(u)$ | |
| $\pi_n(U^{X \mid Y}, u)$ | (136) | $\pi_n((X \mid Y)^U, u)$ | |
| $\mathrm{Ob}_f, \mathbf{C}_c, \mathbf{C}_{cf}$ | (7, 99) | $\mathrm{Ob}_c, \mathbf{F}_f, \mathbf{F}_{fc}$ | |
| $\mathrm{Ho}\,(\mathbf{C}) = \mathbf{C}_{cf}/\simeq$ | (99) | $\mathrm{Ho}\,(\mathbf{F}) = \mathbf{F}_{fc}/\simeq$ | |
| $R: \mathbf{C}_c \to \mathbf{C}_{cf}/\simeq$ | (100) | $M: \mathbf{F}_f \to \mathbf{F}_{fc}/\simeq$ | |

## (iv) Further symbols in cofibration categories