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I

Axioms for homotopy theory and examples of cofibration categories

Axiomatic homotopy theory is the development of the basic constructions of homotopy theory in an abstract setting, so that they may be applied to other categories. And there is, indeed, a strikingly wide variety of categories where these techniques are useful (e.g. topological spaces, differential algebras, differential Lie algebras, chain complexes, modules, sheaves, local algebras. . . .).

The subject is not new and goes back, for example to Kan (1955), Quillen (1967), Heller (1968), and K.S. Brown (1973) each of whom proposes a different set of axioms. In fact, it is not evident what is the most appropriate choice. The best-known approach is that of Quillen who introduces the notion of a (closed) model category, as the starting point for his development of the quite sophisticated 'homotopical algebra'.

The set of axioms which define a model category is, however, quite restrictive. For instance, they do not apply to topological spaces with the usual definitions of fibrations and cofibrations. There are other examples, as pointed out by K.S. Brown, where they give rise to a 'somewhat unsatisfactory' homotopy theory.

We here introduce the notion of a *cofibration category*. Its defining axioms have been chosen according to two criteria:

(1) The axioms should be sufficiently strong to permit the basic constructions of homotopy theory.
(2) The axioms should be as weak (and as simple) as possible, so that the constructions of homotopy theory are available in as many contexts as possible.

They are substantially weaker than the axioms of Quillen, but add one essential axiom to those of Brown. In this chapter we compare the various

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systems of axioms in the literature. It turns out that if one applies the criteria above to these axioms one is almost forced into the definition of a cofibration category. In the chapters to follow we present some of the homotopy theory which can be derived from the axioms of a cofibration category.

In this chapter we also describe many examples of cofibration categories and of fibration categories. In an introductory section §0 we recall the classical definitions of fibrations and cofibrations in topology. Using the notion of an I-category (I = cylinder functor) we prove that the cofibrations in topology satisfy the axioms of a cofibration category. A strictly dual proof shows that fibrations in topology satisfy the axioms of a fibration category. Moreover, we give a complete proof that the algebraic categories of chain complexes, chain algebras, commutative cochain algebras, and chain Lie algebras respectively satisfy the axioms of a cofibration category.

§0 Cofibrations and fibrations in topology

Cofibrations and fibrations are of fundamental importance in homotopy theory. Here we recall their mutual dual definitions which imply many properties which correspond to each other. We will deduce such properties from the axioms of a cofibration category, see §1. Hence the theory of topological cofibrations and fibrations has two aspects:

(1) The study of all properties which can be derived from the axioms (this is part of homotopical algebra).
(2) The study of properties which are highly connected with the topology, for example local properties as studied in tom Dieck–Kamps–Puppe (1970) or James (1984).

In textbooks on algebraic topology and homotopy theory these two aspects are often mixed. This creates unnecessary complexity. Using the axioms we will see that many results on fibrations and cofibrations, respectively, deserve only one proof.

Let **Top** be the **category of topological spaces** and of continuous maps and let

$$I = [0,1] \subset \mathbb{R}$$

be the unit interval of real numbers. These data are the basis of usual homotopy theory. The notion of homotopy can be introduced in two different ways, by use of a cylinder, or by use of a path space:

The **cylinder** I is the functor

$$(0.1)\qquad \begin{cases} I:\mathbf{Top} \to \mathbf{Top}, \\ IX = I \times X \text{ with product topology,} \end{cases}$$

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for which we have the 'structure maps'

$$X \xrightarrow{i_0, i_1} IX \xrightarrow{p} X$$

with $i_0(x) = (0, x)$, $i_1(x) = (1, x)$, $p(t, x) = x$ $(t \in I, x \in X)$. These maps are natural with respect to maps $X \to Y$ in **Top**. The **path space** P is the functor

(0.2) $$\begin{cases} P: \mathbf{Top} \to \mathbf{Top}, \\ PX = X^I, \end{cases}$$

where X^I is the set of all maps $\sigma: I \to X$ with the compact open topology. Now we have the 'structure maps'

$$X \xrightarrow{i} X^I \xrightarrow{q_0, q_1} X$$

with $i(x)(t) = x$ and $q_0(\sigma) = \sigma(0)$, $q_1(\sigma) = \sigma(1)$.

The product topology for IX and the compact open topology for $PY = Y^I$ have the well-known property that a map $G: IX \to Y$ is continuous if and only if the **adjoint map** $\bar{G}: X \to Y^I$ with $\bar{G}(x)(t) = G(t, x)$ is continuous. Therefore we have the bijection of sets

(0.3) $$\mathbf{Top}(IX, Y) = \mathbf{Top}(X, Y^I),$$

which carries G to $\bar{G}$. Here **Top** (A, B) denotes the set of all maps $A \to B$ in **Top**. The bijection shows that the following two definitions of homotopy are equivalent;

(0.4) ***Definition.*** Maps $f_0, f_1: X \to Y$ are **homotopic** $(f_0 \simeq f_1)$ if there is a map $G: IX \to Y$ with $Gi_0 = f_0, Gi_1 = f_1$. ||

(0.5) ***Definition.*** Maps $f_0, f_1: X \to Y$ are **homotopic** $(f_0 \simeq f_1)$ if there is a map $H: X \to PY$ with $q_0 H = f_0, q_1 H = f_1$. ||

There is a standard proof that the relation of homotopy $\simeq$ is an equivalence relation on $\mathbf{Top}(X, Y)$. Moreover, this relation is compatible with the law of composition in **Top**, that is, for maps $f_i: X \to Y$, $g_i: Y \to Z$ $(i = 0, 1)$ with $f_0 \simeq f_1$, $g_0 \simeq g_1$ we get $g_0 f_0 \simeq g_1 f_1$. Therefore the **homotopy category** ***Top***/ $\simeq$ is defined. The morphisms are the homotopy classes of maps in **Top**. The set of morphism in $\mathbf{Top}/\simeq$ from X to Y is the set

(0.6) $$[X, Y] = \mathbf{Top}(X, Y)/\simeq$$

of homotopy classes. For a map $f: X \to Y$ let $\{f\} \in [X, Y]$ be the homotopy class represented by f, we also write $\{f\}: X \to Y$.

(0.7) ***Definition.*** A map $f: X \to Y$ in **Top** is a **homotopy equivalence** if (a) or

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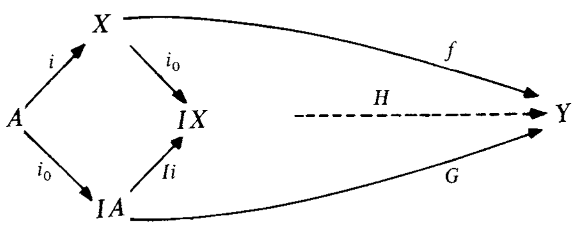
equivalently (b) is satisfied:

(a) $\{f\}$ is an isomorphism in **Top**$/\simeq$.
(b) There is a map $g: Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$. ||

Next we introduce cofibrations and fibrations in **Top** by use of the cylinder and the path space respectively.

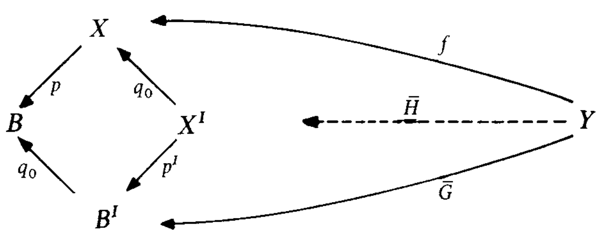
(0.8) ***Definition.*** A map $i: A \to X$ has the **homotopy extension property** (HEP) with respect to Y if for each commutative diagram of unbroken arrows in **Top**



there exists H extending the diagram commutatively. The map i is a **cofibration** in **Top** if it has the homotopy extension property with respect to any space in **Top**. The cofibration i is **closed** if iA is a closed subspace . ||

The following definition of a fibration is **dual** to the definition of a cofibration in the sense that the cylinder is replaced by the path space and all arrows are replaced by arrows in the opposite direction.

(0.9) ***Definition.*** A map $p: X \to B$ has the **homotopy lifting property** (HLP) with respect to Y if for each diagram of unbroken arrows in **Top**



there exists $\bar{H}$ extending the diagram commutatively. The map p is a **fibration** in **Top** if it has the homotopy lifting property with respect to any space in **Top**. ||

Using the adjunction (0.3) we can reformulate this definition as follows: The map p has the HLP with respect to Y iff for each commutative diagram

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of unbroken arrows

(0.10)
$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle i_0}\downarrow & \overset{H}{\nearrow} & \downarrow \\ IY & \xrightarrow[G]{} & B \end{array}$$

there is H extending the diagram commutatively. The map H is called a **lifting** of G.

Let $D^n = \{x \in \mathbb{R}^n;\ \|x\| \leqq 1\}$ be the **disk** in $\mathbb{R}^n$ with boundary $\partial D^n = S^{n-1}$, $n \geqq 0$. For $n = 0$ we have $S^{-1} = \varnothing =$ the empty set.

(0.11) ***Definition.*** A map $p: X \to B$ is a **Serre-fibration** if p has the HLP with respect to D^n, $n \geqq 0$. ||

On the other hand, we obtain by use of the disks the following cofibrations in **Top**.

(0.12) ***Definition.*** We say that $A \subset X$ is given by **attaching a cell** to A if there exists a push out diagram in **Top** ($n \geqq 0$)

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ \cup & & \cup \\ S^{n-1} & \longrightarrow & A \end{array}$$

The inclusions $S^{n-1} \subset D^n$ and $A \subset X$ are cofibrations in **Top**. ||

In the next section we define a cofibration category. A basic example of a cofibration category is the category **Top** with cofibrations as in (0.8) and with weak equivalences given by homotopy equivalences in **Top**, compare (5.1) below. Moreover, we will see that fibrations and homotopy equivalences in **Top** satisfy the axioms of a fibration category which are obtained by dualizing the axioms of a cofibration category, see (1a.1) and (5.2) below.

§1 Cofibration categories

Here we introduce the notion of a cofibration category. This is a category together with two classes of morphisms, called cofibrations and weak equivalences, such that four axioms $C1, \dots, C4$ are satisfied.

(1.1) ***Definition.*** A **cofibration category** is a category **C** with an additional structure

$$(\mathbf{C}, cof, we),$$

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subject to axioms C1, C2, C3 and C4. Here *cof* and *we* are classes of morphisms in **C**, called **cofibrations** and **weak equivalences** respectively. ‖

Morphisms in **C** are also called *maps* in **C**. We write $i: B \subset A$ or $B \rightarrowtail A$ for a cofibration and we call $u|B = ui: B \to U$ the **restriction** of $u: A \to U$. We write $X \xrightarrow{\sim} Y$ for a weak equivalence in **C**. An isomorphism in **C** is denoted by $\cong$. The identity of the object X is $1 = 1_X = id$. A map in **C** is a **trivial cofibration** if it is both a weak equivalence and a cofibration. An object R in a cofibration category **C** will be called a **fibrant model** (or simply **fibrant**) if each trivial cofibration $i: R \overset{\sim}{\rightarrowtail} Q$ in **C** admits a retraction $r: Q \to R$, $ri = 1_R$.

The axioms in question are:

(C1) *Composition axiom*: The isomorphisms in **C** are weak equivalences and are also cofibrations. For two maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

if any two of f, g, and gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

(C2) *Push out axiom*: For a cofibration $i: B \rightarrowtail A$ and map $f: B \to Y$ there exists the push out in **C**

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A \bigcup_B Y = A \bigcup_f Y \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \bar{i}} \\ B & \xrightarrow{f} & Y \end{array}$$

and $\bar{i}$ is a cofibration. Moreover:

(a) if f is a weak equivalence, so is $\bar{f}$,
(b) if i is a weak equivalence, so is $\bar{i}$.

(C3) *Factorization axiom*: For a map $f: B \to Y$ in **C** there exists a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle g}\,\sim & \\ & A & \end{array}$$

where i is a cofibration and g is a weak equivalence.

(C4) *Axiom on fibrant models*: For each object X in **C** there is a trivial cofibration $X \overset{\sim}{\rightarrowtail} RX$ where RX is fibrant in **C**. We call $X \overset{\sim}{\rightarrowtail} RX$ a **fibrant model of** X.

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(1.2) ***Remark.*** We denote by Ob_f a class of fibrant models in **C** which is **sufficiently large**, this means that each object in **C** has a fibrant model in Ob_f. Let $\mathbf{C}_f$ be the full subcategory of **C** with objects in Ob_f. By the structure of **C** we have cofibrations and weak equivalences in $\mathbf{C}_f$. One can check that $\mathbf{C}_f$ satisfies the axioms (C1), (C3) and (C4) but not necessarily axiom (C2) since the push out of objects in Ob_f needs not to be an object in Ob_f. If, however, push outs as in (C2) exist in $\mathbf{C}_f$ then $\mathbf{C}_f$ is a cofibration category in which all objects are fibrant.

(1.3) ***Remark.*** Let ϕ be an **initial object** of the cofibration category **C**. We call an object X in **C** **cofibrant** if $\phi \to X$ is a cofibration. Let $\mathbf{C}_c$ be the full subcategory of C consisting of cofibrant objects. By the structure of **C** we have cofibrations and weak equivalences in the category $\mathbf{C}_c$. One easily checks the axioms (C1), ..., (C4). Thus $\mathbf{C}_c$ is a cofibration category in which all objects are cofibrant. We point out that the notion 'cofibrant' is not dual to the notion 'fibrant' in (C4). Therefore we call a cofibrant object in **C** as well a **ϕ-cofibrant** object since its definition depends on the existence of the initial object ϕ.

The development of the homotopy theory in a cofibration category is most convenient if all objects in **C** are fibrant and cofibrant.

(1.4) **Lemma.** *Let* **C** *be a cofibration category. Then* (C2) (a), (C1) *and* (C3) *imply* (C2) (b). *If all objects in* **C** *are cofibrant then* (C2) (b), (C1) *and* (C3) *imply* (C2) (a).

Thus axiom (C2) (b) is redundant. We call (C2)(a) the '*axiom of properness*' (compare (2.1) below); many results in a cofibration category actually do not rely on this axiom. If all objects are cofibrant then the axiom of properness is redundant by (1.4).

Proof. We consider the push out diagrams

$$\begin{array}{ccccc} A & \overset{\bar{j}}{\rightarrowtail} & \bar{X} & \xrightarrow{\bar{g}} & \bar{Y} \\ {\scriptstyle i}\uparrow & \text{push} & {\scriptstyle i_1}\uparrow & \text{push} & \uparrow{\scriptstyle \bar{i}}, \\ B & \underset{j}{\rightarrowtail} & X & \underset{g}{\xrightarrow{\sim}} & Y \end{array}$$

were $g j = f$ by (C3). If i is a weak equivalence, so is i_1 *by* (C2) (a). Moreover, since g is a weak equivalence, also $\bar{g}$ is one by (C2) (a). Thus by (C1) also $\bar{i}$ is a weak equivalence.

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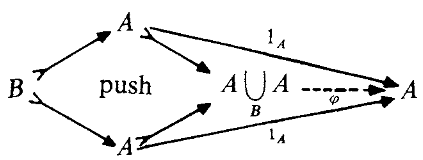
For the proof of the second part of (1.4) we need the mapping cylinders in (1.8) below. We continue the proof of (1.4) in the appendix §1b of this section. □

We define cylinders and the notion of homotopy in a cofibration category as follows: Let $B \rightarrowtail A$ be a cofibration. Then we have by (C2) the push out diagram



where $\varphi = (1_A, 1_A)$ is called the **folding map**. By (C3) there is a factorization

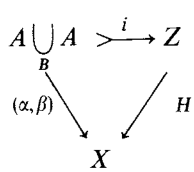
(1.5) $$A \bigcup_B A \overset{i}{\rightarrowtail} Z \underset{\sim}{\overset{p}{\to}} A$$

of the folding map φ. We call $Z = I_B A$ together with i and p in (1.5) a **relative cylinder** on $B \rightarrowtail A$. For $i = (i_0, i_1)$ the maps $i_\varepsilon: A \overset{\sim}{\rightarrowtail} Z$ are trivial cofibrations since $pi_\varepsilon = 1_A$; use (C1).

Let X be a fibrant object.

Two maps $\alpha, \beta: A \to X$ are **homotopic relative** B (or **under** B) and we write $\alpha \simeq \beta$ rel B if there is a commutative diagram

(1.6)



where Z is a relative cylinder on $B \rightarrowtail A$. We call H a **homotopy from** α **to** β **rel** B. We will prove that homotopy rel B is an equivalence relation, see Chapter II.

For a ϕ-cofibrant object A there exists the sum $A + Y$ (also denoted by $A \vee Y$). The sum is given via (C2) by the push out

(1.7) $$A + Y = A \bigcup_\phi Y = A \vee Y$$

where $Y \rightarrowtail A + Y$ is a cofibration by (C2). Also $A \rightarrowtail A + Y$ is a cofibration provided Y is ϕ-cofibrant. We define the **mapping cylinder** Z_f of $f: A \to Y$ by a factorization of the map $(1_Y, f): Y + A \to Y$ via (C3):

(1.8) $$(1_Y, f): Y + A \rightarrowtail Z_f \underset{q}{\overset{\sim}{\to}} Y.$$

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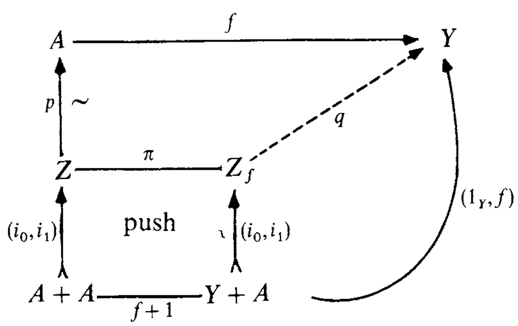
If Y is cofibrant this yields the factorization $f = qi_1$,

$$f: A \underset{i_1}{\rightarrowtail} Z_f \underset{i_0}{\overset{q}{\rightleftarrows}} Y, \tag{a}$$

where q is a retraction of $i_0: Y \rightarrowtail Y + A \rightarrowtail Z_f$ and where $i_1: A \rightarrowtail Y + A \rightarrowtail Z_f$. Moreover, we can use the cylinder $Z = I_\phi A$ in (1.5) for the construction of the mapping cylinder via a push out diagram:



(b)

Here i_0 is a weak equivalence since $i_0: A \rightarrowtail Z$ is a weak equivalence. Therefore the retraction q of i_0 is a weak equivalence by (C1).

(1.9) ***Definition.*** A commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\uparrow & \simeq \text{push} & \uparrow \\ B & \underset{g}{\longrightarrow} & D \end{array}$$

in a cofibration category **C** is a **homotopy push out** (or **homotopy cocartesian**) if for some factorization $B \rightarrowtail W \xrightarrow{\sim} A$ of f the induced map

$$W \bigcup_B D \to C$$

is a weak equivalence. This easily implies that for any factorization $B \rightarrowtail V \xrightarrow{\sim} A$ of f, the map $V \bigcup_B D \to C$ is a weak equivalence. Thus in the definition we could have replaced 'some' by 'any' or used g in place of f. We leave the proof of these remarks as an exercise, compare (II,§ 1) ||

Next we consider functors between cofibration categories.

(1.10) ***Definition.*** Let **C** and **K** be cofibration categories and let $\alpha: \mathbf{C} \to \mathbf{K}$ be a functor.

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(1) The functor α is **based** if **C** and **K** have an initial object (denoted by $*$) with $\alpha(*) = *$.

(2) The functor α **preserves weak equivalences** if α carries a weak equivalence in **C** to a weak equivalence in **K**.

(3) Let

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & Y = A \bigcup\limits_B X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ B & \longrightarrow & X \end{array}$$

be a push out diagram in **C**. We say that α is **compatible with the push out** $A \bigcup_B X$ if the induced diagram

$$\begin{array}{ccc} \alpha A & \longrightarrow & \alpha(A \bigcup\limits_B X) \\ \uparrow & & \uparrow \\ \alpha B & \longrightarrow & \alpha X \end{array}$$

is a homotopy push out in **K**, see (1.9).

(4) We call α a **model functor** if α preserves weak equivalences and if α is compatible with all push outs as in (3). Hence a model functor α carries homotopy cocartesian diagrams in **C** to homotopy cocartesian diagrams in **K**. ||

We will see that a based model functor is compatible with most of the constructions in a cofibration category described in this book. In general, we do not assume that a model functor carries a cofibration in **C** to a cofibration in **K**.

§ 1a Appendix: fibration categories

By dualizing (1.1) we obtain

(1a.1) ***Definition.*** A **fibration category** is a category **F** with the structure

$$(\mathbf{F}, fib, we),$$

subject to axioms (F1), (F2), (F3) and (F4). Here *fib* and *we* are classes of morphisms in **F**, called **fibrations** and **weak equivalences** respectively. These morphisms satisfy the condition that the **opposite category** $\mathbf{C} = \mathbf{F}^{op}$ is a

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