ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Combinatorial Geometries

Edited by

NEIL WHITE

University of Florida

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York Port Chester Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1987
Reprinted 1990

British Library cataloguing in publication data

Combinatorial geometries. – (Encyclopedia of mathematics and its applications; v. 29)

1. Combinatorial geometry
I. White, Neil
516'.13 QA167

Library of Congress cataloguing in publication data

Combinatorial geometries.

(Encyclopedia of mathematics and its applications; v. 29)
Includes bibliographies and index.
1. Matroids. 2. Combinatorial geometry. I. White, Neil. II. Series.
QA166.6.C66 1987 516'.13 86–31001

ISBN 0 521 33339 3

Transferred to digital printing 2003

TM

CONTENTS

1

Coordinatizations

NEIL WHITE

1.1. Introduction and Basic Definitions

The purpose of this chapter is to provide background and general results concerning coordinatizations, while the more specialized subtopics of binary and unimodular matroids are covered in later chapters. The first section of this chapter is devoted to definitions and notational conventions. The second section concerns linear and projective equivalence of coordinatizations. Although they are not usually explicitly considered in other expositions of matroid coordinatization, these equivalence relations are very useful in working with examples of coordinatizations, as well as theoretically useful as in Proposition 1.2.5. Section 1.3 involves the preservation of coordinatizability under certain standard matroid operations, including duality and minors. The next section presents some well-known counterexamples, and Section 1.5 considers characterizations of coordinatizability, especially characterizations by excluded minors. The final five sections are somewhat more technical in nature, and may be omitted by the reader who desires only an introductory survey. Section 1.6 concerns the bracket conditions, another general characterization of coordinatizability. Section 1.7 presents techniques for construction of a matroid requiring a root of any prescribed polynomial in a field over which we wish to coordinatize it. These techniques are extremely useful in the construction of examples and counterexamples, yet are not readily available in other works, except Greene (1971). The last three sections concern characteristic sets, the use of transcendentals in coordinatizations, and algebraic representation (i.e., modeling matroid dependence by algebraic dependence). Some additional topics which could have been considered here, such as chain groups, are omitted because they are well-covered in other readily available sources, such as Welsh (1976).

Since the prototypical example of a matroid is an arbitrary subset of a finite dimensional vector space, that is, a vector matroid, and since many matroid operations have analogs for vector spaces, which are algebraic and therefore easier to employ, a natural and important problem is to determine which matroids are isomorphic to vector matroids. This leads directly to the concept of coordinatization. In this chapter we assume that matroids are finite.

A *coordinatization* of a matroid $M(S)$ in a vector space V is a mapping $\zeta: S \to V$ such that for any $A \subseteq S$, A is independent in $M \Leftrightarrow \zeta|_A$ is injective (one-to-one) and $\zeta(A)$ is linearly independent in V.

Thus we note that a dependent set in M may either be mapped to a linearly dependent set in V or mapped non-injectively.

We note that $\zeta(s)=0$ if and only if s is a loop. Moreover for non-loops s and t, $\zeta(s)$ is a non-zero scalar multiple of $\zeta(t)$ if and only if $\{s,t\}$ is a circuit (i.e., s and t are parallel). Thus $\zeta(s)=\zeta(t)$ only if $\{s,t\}$ is a circuit, and we see that non-injective coordinatizations exist only for matroids which are not combinatorial geometries. Furthermore, we also see that coordinatizing a matroid is essentially equivalent to coordinatizing its associated combinatorial geometry.

If B is any basis of $M(S)$, then let W be the span of $\zeta(B)$ in V. Then dim $W = \text{rk } M$ and $\zeta(S) \subseteq W$. Thus we may restrict the range of ζ to W, and thus, without loss of generality, all coordinatizations will be assumed to be in a vector space of dimension equal to the rank of the matroid. If n is the rank of $M(S)$, then for a given field K there is, up to isomorphism, a unique vector space V of dimension n over K. Thus we may also speak of a *coordinatization of M over K*, meaning a coordinatization in V.

Let $GF(q)$ denote the finite field of order q. A matroid which has a coordinatization over $GF(2)$, or $GF(3)$, is called *binary*, or *ternary*, respectively. A matroid which may be coordinatized over every field is called *unimodular* (or *regular*). Further characterizations of these classes of matroids will be given later in this chapter and in the following chapters.

It is often convenient to represent a coordinatization in matrix form. If $\zeta: S \to V$ is a coordinatization of $M(S)$ of rank n, and E a basis of V, let $A_{\zeta,E}$ be the matrix with n rows and with columns indexed by S whose a-th column, for $a \in S$, is the vector $\zeta(a)$ represented with respect to E. Since the matrix $A_{\zeta,E}$ also determines the coordinatization ζ if we are given E, we often simply say $A_{\zeta,E}$ is a coordinatization of $M(S)$.

1.2. Equivalence of Coordinatizations and Canonical Forms

If $\phi: V \to V$ is a non-singular linear transformation and $\zeta: S \to V$ is a coordinatization of $M(S)$, then $\phi \circ \zeta: S \to V$ is also a coordinatization. If Q is the non-singular $n \times n$ matrix representing ϕ with respect to the basis E of V, then $A_{\phi\circ\zeta,E} = QA_{\zeta,E}$. On the other hand, we may easily check that

$A_{\phi\circ\zeta,E} = A_{\zeta,\phi^{-1}E}$, so multiplying $A_{\zeta,E}$ on the left by Q may also be regarded as simply a change of basis for the coordinatization ζ.

We recall from elementary linear algebra that multiplying $A_{\zeta,E}$ on the left by a non-singular matrix Q is equivalent to performing a sequence of elementary row operations on $A_{\zeta,E}$, and that any such sequence of elementary row operations on $A_{\zeta,E}$ may be realized by an appropriate choice of Q. We will say $A_{\zeta,E}$ and $QA_{\zeta,E}$ are *linearly equivalent* (where Q is non-singular), and any matrix linearly equivalent to $A_{\zeta,E}$ may be regarded as representing the same coordinatization ζ of the same matroid with respect to a new basis of V.

Conversely, given a coordinatization matrix $A_{\zeta,E}$, we may choose any new basis E' of V, and $A_{\zeta,E'}$ is linearly equivalent to $A_{\zeta,E}$. As a special case of this, we pick $E' = \zeta(B)$, where B is a fixed basis of the matroid $M(S)$.

Then, by reordering the elements of S so that the first n elements are the elements of B, we have a matrix $A_{\zeta,E'}$ in echelon form

$$\begin{array}{cc} & \quad B \quad S-B \\ A_{\zeta,E'} = & \left(I_n \middle| \quad L\right) \end{array}$$

where I_n is the $n \times n$ identity matrix, with columns indexed by B, and L is an $n \times (N-n)$ matrix with columns indexed by $S - B$, where $N = |S|$.

As yet another way of viewing linear equivalence, let W_ζ be the subspace spanned by the rows of $A_{\zeta,E'}$ in an N-dimensional vector space U. What we have seen is that W_ζ is independent of E', and that indeed the choice of E' actually amounts to a choice of a basis for W_ζ. Thus every linear equivalence class of $n \times N$ matrices coordinatizing $M(S)$ corresponds to an n-dimensional subspace of U. Conversely, every n-dimensional subspace of U corresponds to a coordinatization of some rank n matroid on S, which is a weak-map image of $M(S)$.

Remark. Algebraic geometers regard the collection of all n-dimensional subspaces of an N-dimensional vector space as a Grassmann manifold, and the coordinatizations of $M(S)$ correspond to a certain submanifold.

Besides row operations, another operation on $A_{\zeta,E}$ which leaves invariant the matroid coordinatized by $A_{\zeta,E}$ is non-zero scalar multiplication of columns. This may be accomplished by multiplying $A_{\zeta,E}$ on the right by an $N \times N$ diagonal matrix with non-zero diagonal entries. Combining this with the previous operations, we say that two $n \times N$ matrices A and A' are *projectively equivalent* if there exist Q, an $n \times n$ non-singular matrix, and D, an $N \times N$ non-singular diagonal matrix, such that $A' = QAD$.

Let us recall that projective $n-1$ dimensional space P is obtained from V by identifying the non-zero vectors of each one-dimensional subspace of V to give a point of P. Let $\pi: V \to P \cup \{0\}$ be the resulting map, where 0 is an element adjoined to P which is the image of $0 \in V$. Then if $\zeta: S \to V$ is a coordinatization,

$\pi \circ \zeta$ is an embedding of $M(S)$ into $P \cup \{0\}$, except that parallel elements become identified in $P \cup \{0\}$. If $T': V \to V$ is a linear transformation, let $T = \pi \circ T' \circ \pi^{-1}$, which is well-defined since T' preserves scalar multiples. Then we call T a linear transformation of $P \cup \{0\}$. Since non-zero scalar multiples in V are identified in $P \cup \{0\}$, we immediately have the following:

1.2.1. Proposition. *Let J and L be $n \times N$ matrices over the field K. Then if J coordinatizes $M(S)$ and J is projectively equivalent to L, then L also coordinatizes $M(S)$. J and L are projectively equivalent if and only if their corresponding coordinatizations ζ_J and ζ_L determine the same projective embedding up to change of basis in $P \cup \{0\}$, i.e., $\pi \circ \zeta_J = T \circ \pi \circ \zeta_L$, where T is a non-singular linear transformation of $P \cup \{0\}$.*

We next ask whether there exists a canonical form for a projective equivalence class of coordinatizations, as echelon form was for a linear equivalence class. For a given coordinatization

$$A = (I_n | L)$$

in echelon form with respect to a basis B, let L^+ be the matrix obtained by replacing each non-zero entry of L by 1. In fact, L^+ is just the incidence matrix of the elements of B with the basic circuits of the elements of $S - B$, so it is independent of the particular coordinatization. Now let Γ be the bipartite graph whose adjacency matrix is L^+. Thus each entry of 1 in L^+ corresponds to an edge of Γ. Let T be a basis (i.e., spanning tree) of Γ.

1.2.2. Proposition. (*Brylawski and Lucas, 1973*) *A is projectively equivalent to a matrix A' which is in echelon form with respect to B, and which has 1 for each entry corresponding to an edge of T.*

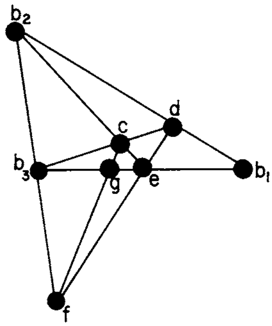
Proof. This may be accomplished by non-zero scalar multiplication of rows and columns, and is left as an exercise. □

The matrix A' of the preceding proposition is said to be in (B, T)-*canonical form*, or when B and T are understood, *canonical projective form*. The simplest canonical projective form and most useful version of this canonical form occurs when $M(S)$ has a spanning circuit C. Then by choosing B to be $C - \{c\}$ for some $c \in C$, the column corresponding to c in L has no zeros, hence we may pick T to correspond to the n entries of column c, together with the first non-zero entry in every other column of L.

A major use of this projective canonical form is in actual computation with coordinates and in presenting examples.

1.2.3. Example. Let $M(S)$ be the 8-point rank 3 geometry whose affine diagram appears in Figure 1.1. If we choose the standard basis $B = \{b_1, b_2, b_3\}$

Figure 1.1. An 8-point rank 3 geometry.



and spanning circuit $C = \{b_1, b_2, b_3, c\}$, we may coordinatize M over $\mathbb{Q}$ by the following matrix in canonical projective form:

$$\begin{array}{cccccccc} b_1 & b_2 & b_3 & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}.$$

1.2.4. Example. Let $M(S)$ be the 4-point line, that is, $U_{2,4}$, the uniform geometry of cardinality 4 and rank 2, whose bases are all of the subsets of S of cardinality 2, where $|S| = 4$. Then any coordinatization of $M(S)$ over any field K may be put in the following projective echelon form:

$$\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{matrix}$$

where $\alpha \in K - \{0, 1\}$. Thus we can say that up to projective equivalence, there is a one-parameter family of coordinatizations of $U_{2,4}$. We note that this parameter α is equivalent to the classical cross-ratio of four collinear points in projective geometry.

Since $U_{2,4}$ is the simplest non-binary matroid, one might be led to surmise the following, first proved by White (1971, Proposition 5.2.5), and later by Brylawski and Lucas (1973) using more elementary techniques. The proof is omitted here, because of its fairly technical nature.

1.2.5. Proposition. *Let $M(S)$ be a binary matroid and K a field over which M has a coordinatization. Then any two coordinatizations of M over K are projectively equivalent.*

Brylawski and Lucas (1973) have investigated the question of which matroids have, over a particular field K, any two coordinatizations projectively equivalent. Such matroids are said to be *uniquely coordinatizable over K*, and among their findings is that ternary matroids are uniquely coordinatizable over $GF(3)$ (although not over an arbitrary field, as the example of $U_{4,2}$ shows).

1.2.6. Example. We return to Example 1.2.3. This example is, in fact, a ternary matroid, which is uniquely coordinatizable not only over $GF(3)$, but over every field K such that char $K \neq 2$. To see this, we first note that the matrix given over $\mathbb{Q}$ may be regarded as a coordinatization of M over every field K such that char $K \neq 2$. If we take an arbitrary coordinatization of M over any such field K and put that coordinatization in canonical projective form with respect to B and C, the elements b_1, b_2, b_3, and c are assigned the vectors shown, and then the vector for d is determined since d is on the intersection of the two lines b_1b_2 and b_3c. Likewise $e \in b_1b_3 \cap b_2c$, $f \in b_2b_3 \cap de$, and $g \in b_1b_3 \cap cf$.

1.3. Matroid Operations

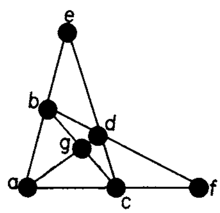
We now note that coordinatizability is preserved under various matroid operations, including duality, minors, direct sums, and, in a restricted sense, truncation. This material is also found scattered through Chapter 7 of White (1986), and is collected here for convenience.

1.3.1. Proposition. *Let $A_{\zeta,E}$ coordinatize $M(S)$ over a field K, and let W_ζ be the row-space of $A_{\zeta,E}$ in U, a vector space of dimension $N = |S|$ over K. Then if $M^*(S)$ denotes the dual matroid of M, the subspace $W_\zeta^\perp$ orthogonal to W_ζ is the subspace of U corresponding to a coordinatization of M^*. Thus M is coordinatizable over K if and only if M^* is.*

Furthermore, if $A_{\zeta,E}$ is in echelon form, $A_{\zeta,E} = (I_n, L)$, then $A^* = (-L^t, I_{N-n})$ is a coordinatization of M^*, where t denotes transpose.

Proof. Let B be a basis of $M(S)$ and we may assume $A_{\zeta,E}$ is in echelon form with respect to B, since W_ζ is invariant under linear equivalence. Thus $A_{\zeta,E} = (I_n, L)$, and we note that $A^* = (-L^t, I_{N-n})$ has each of its rows orthogonal to each row of $A_{\zeta,E}$, hence the rows of A^* are a basis of $W_\zeta^\perp$. Let $M'(S)$ be the matroid coordinatized by the columns of A^*. Since $S - B$ corresponds to the columns

Figure 1.2. A 7-point rank 3 matroid M.



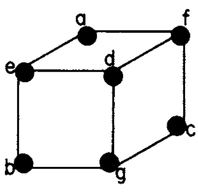
of I_{N-n} in A^*, we see that $S - B$ is a basis of M'. Conversely, if B' is any basis of M', $S - B'$ is a basis of M by a similar argument. Since B was an arbitrary basis of M, $M' = M^*$ and the theorem follows. □

1.3.2. Example. Let $M(S)$ be the 7-point rank 3 matroid shown in Figure 1.2, along with a coordinatization A over $\mathbb{R}$ given below. Then M^*, a rank 4 matroid which is shown in Figure 1.3, has the coordinatization A^* over $\mathbb{R}$ as in the preceding proposition.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad (\text{columns } a\ b\ c\ d\ e\ f\ g)$$

$$A^* = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{columns } a\ b\ c\ d\ e\ f\ g)$$

Figure 1.3. M^*, the dual of the matroid M in Figure 1.2, where *abfg*, *aceg*, *bcef* are coplanar sets.



1.3.3. Proposition. *Let $M(S)$ be a matroid.*

(1) *If M is coordinatizable over a field K, then so is every minor of M.*

(2) *If $M = M_1 \oplus M_2$, then M is coordinatizable over K if and only if both M_1 and M_2 are coordinatizable over K.*

(3) *If K is sufficiently large and M is coordinatizable over K, then the truncation $T(M)$ is coordinatizable over K.*

Proof. (1) If $A_{\zeta,E}$ coordinatizes M, then any submatroid $M - X$ is coordinatized by deleting the columns of $A_{\zeta,E}$ corresponding to X. Since contraction is the dual operation to deletion, (1) follows from the preceding proposition. For a direct construction of a coordinatization of a contraction, see the following remark and example.

(2) If $A^{(1)}$ and $A^{(2)}$ are matrices coordinatizing M_1 and M_2 respectively, then

the matrix direct sum

$$\begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix}$$

is a coordinatization of $M = M_1 \oplus M_2$. The converse follows from (1).

(3) The construction of truncation (to rank $n-1$, say) described in Section 7.4 of White (1986) may be carried out within the vector space V provided only that the field is sufficiently large to guarantee the existence of a free extension (by one point) within V. □

1.3.4. Remark. To construct the coordinatization of a contraction $M(S)/X$ from a coordinatization $A_{\zeta,E}$ of M, we first choose a basis I of the set X. By row operations on $A_{\zeta,E}$ we may make the first $n-k$ entries 0 in each column corresponding to I, where $k = |I|$. Then delete the columns corresponding to X, as well as the last k rows.

This construction really amounts to simply taking a linear transformation T from V, the vector space in which M is coordinatized, to a vector space of dimension $n-k$, such that the kernel of T is precisely span (ζX).

1.3.5. Example. Let M be the matroid shown in Figure 1.4, with coordinatization A over $\mathbb{Q}$. Let $X = \{e, f\}$. Then row operations on A lead to the matrix A', and deletion of the appropriate rows and columns gives A'', a coordinatization of M/X, which is put into canonical projective form A'''. The matroid M/X is shown in Figure 1.5.

$$A = \begin{pmatrix} \overset{a}{1} & \overset{b}{0} & \overset{c}{0} & \overset{d}{0} & \overset{e}{1} & \overset{f}{1} & \overset{g}{1} & \overset{h}{0} \\ 0 & 1 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 7 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -5 \end{pmatrix},$$

$$A' = \begin{pmatrix} \overset{a}{0} & \overset{b}{0} & \overset{c}{0} & \overset{d}{1} & \overset{e}{0} & \overset{f}{0} & \overset{g}{2} & \overset{h}{-5} \\ -3 & 1 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 7 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$A'' = \begin{pmatrix} \overset{a}{0} & \overset{b}{0} & \overset{c}{0} & \overset{d}{1} & \overset{g}{2} & \overset{h}{-5} \\ -3 & 1 & 0 & 0 & -3 & 0 \end{pmatrix},$$

$$A''' = \begin{pmatrix} \overset{d}{1} & \overset{b}{0} & \overset{g}{1} & \overset{a}{0} & \overset{c}{0} & \overset{h}{1} \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Figure 1.4. A matroid M.

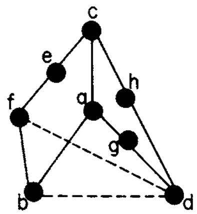
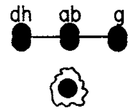


Figure 1.5. M/X, with M as Figure 1.4.

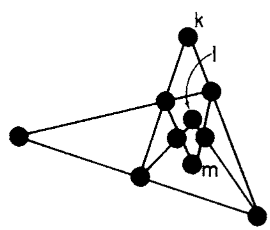


1.4. Non-coordinatizable Geometries

We now give several examples of combinatorial geometries which may not be coordinatized over any field.

The first example is a rank 3 matroid obtained from the Desargues configuration by replacing the 3-point line klm by three 2-point lines, kl, km, and lm, as shown in Figure 1.6. Coordinatization of this matroid over a field K is equivalent to embedding this configuration in the projective plane $P(2, K)$. However, $P(2, K)$ is a Desarguesian plane, which means simply that in this configuration, klm must be collinear, so coordinatization is impossible. This matroid is called the non-Desargues matroid.

Figure 1.6. The non-Desargues matroid.



A second example of a non-coordinatizable geometry, the non-Pappus matroid, is obtained from the Pappus configuration in a manner similar to that just given for the Desargues configuration. This is illustrated in Figure 1.7, where x, y, and z are non-collinear, violating the usual assertion of Pappus' Theorem.

A third example is a class of examples which are the smallest non-coordinatizable geometries in terms of cardinality. The simplest member of this class, discovered by Vámos (1971), is described by letting $S = \{a, b, c, d, a', b', c', d'\}$, and letting the bases of $M(S)$ be all the 4-element

Figure 1.7. The non-Pappus matroid.

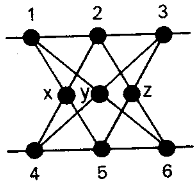
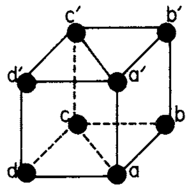


Figure 1.8. A Vámos cube.



subsets of S except $aa'\,bb'$, $bb'\,cc'$, $cc'\,dd'$, $aa'\,dd'$, $aa'\,cc'$. This matroid of rank 4 may be illustrated by the affine diagram in Figure 1.8, even though it cannot actually exist in an affine space as a consequence of its non-coordinatizability.

First we verify that M is actually a combinatorial geometry. This is easy in terms of circuit exchange. The circuits of M are the five 4-element subsets which are not bases, as listed above, together with each 5-element subset of S which does not contain any of the 4-element circuits. Now if C_1 and C_2 are circuits with $C_1 \neq C_2$, and $x \in C_1 \cap C_2$, we first note that $|C_1 \cup C_2| \geqslant 6$, since circuits are incomparable and no two of the 4-element circuits have an intersection of more than two elements. Hence $(C_1 \cup C_2) - x$ has cardinality at least 5, and contains a circuit. Hence $M(S)$ is a geometry.

Next we show that M is, in fact, non-coordinatizable. Suppose, to the contrary, that M has been embedded in $P(3, K)$ for some field K. Then dd', which is not coplanar with $aa'cc'$, must intersect the plane $aa'cc'$ in a point e. But since $e \in aa'dd' \cap cc'dd'$, we must have $e \in aa' \cap cc'$. By a symmetric argument, bb' must also intersect $aa'cc'$ in e, but then b, b', d, and d' are coplanar, contradicting the fact that $bb'dd'$ is a basis of $M(S)$.

Finally we note that further members of this class of examples may be constructed by taking the same set S and the five 4-element circuits given for M, and then listing additional 4-element circuits (and letting all other 4-element subsets of S remain as bases) subject to two constraints:

(i) $bb'dd'$ remains a basis;

(ii) no two of the 4-element circuits intersect in more than two elements.

The argument that the result is a combinatorial geometry which is non-coordinatizable proceeds exactly as above.

The member of this class of examples which has the maximum number of 4-element circuits is the one which has, besides the five given 4-element circuits, *abcd*, *a'b'c'd'*, *abc'd'*, *ab'cd'*, *ab'c'd*, *a'bcd'*, *a'bc'd*, and *a'b'cd*. If *bb'dd'* were also to be made a circuit, the resulting geometry would be isomorphic to a three-dimensional binary affine space, $AG(3,2)$.

The members of a fourth (and very large) class of non-coordinatizable geometries are obtained by taking two geometries G_1 and G_2 such that there is no field over which both G_1 and G_2 may be coordinatized, and then constructing a geometry G_3 which has both G_1 and G_2 as minors. There are many ways of constructing such a geometry G_3, with perhaps the two most natural being the direct sum of G_1 and G_2, and the direct sum truncated to a rank equal to the rank of G_1 or G_2, whichever is larger.

1.5 Necessary and Sufficient Conditions for Coordinatization

The most successful coordinatization conditions are the excluded minor characterizations of the classes of matroids coordinatizable over certain fields. We will discuss these first, and follow with a consideration of conditions for coordinatizability over arbitrary fields.

If A is a class of matroids, an *excluded minor characterization* of A is collection E of matroids with the property that for every matroid M, $M \in A$ if and only if there does not exist $N \in E$ with N isomorphic to a minor of M. Although E could be either finite or infinite, we are primarily interested in this type of characterization when E is finite. It is elementary to check that A has an excluded minor characterization if and only if A is a *hereditary class*, that is, a class of matroids closed under the taking of minors.

The class of binary matroids is by far the best understood class of matroids, because of its particularly simple structure.

1.5.1. Proposition. *A matroid is binary if and only if it has no minor isomorphic to the 4-point line,* $U_{2,4}$.

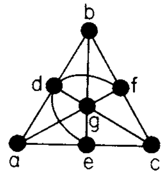
This and many other characterizations of binary matroids are given in Chapter 2.

A particular binary matroid we will frequently refer to is F_7, the Fano plane, given by the following binary coordinatization:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

This matroid is also sometimes referred to as $PG(2,2)$, the projective plane over $GF(2)$, and is illustrated in Figure 1.9.

Figure 1.9. The Fano matroid, F_7.



The excluded minor characterization of ternary matroids was discovered and proved by R. Reid, *c.* 1971, but never published. The result, which follows, was published independently by Bixby (1979) and by Seymour (1979).

1.5.2. Proposition. *A matroid is ternary if and only if it has no minor isomorphic to one of*

$$U_{2,5}, U_{2,5}^* \text{ (which is } U_{3,5}), F_7, \text{ or } F_7^*.$$

A third excluded minor characterization, that of unimodular matroids by Tutte (1958), stands as one of the crowning achievements of matroid theory. This theorem is very deep, as it was first proved by way of Tutte's Homotopy Theorem. There are other proofs now available which are more elementary (Seymour 1979).

1.5.3. Theorem. *A matroid is unimodular if and only if it has no minor isomorphic to one of*

$$U_{2,4}, F_7, F_7^*.$$

Another equally striking characterization of unimodular matroids was found by Seymour (1980). He shows that every unimodular matroid may be built up in certain ways from graphic matroids, cographic matroids, and copies of a particular matroid called R_{10}.

These and several other characterizations of unimodular matroids are discussed in Chapter 3.

There are some very interesting excluded minor characterizations for several classes of graphic matroids. These characterizations are discussed more completely in Chapter 2, but are included here for the sake of completeness.

1.5.4. Theorem. (*Tutte 1959*). *A matroid is graphic if and only if it has no minor isomorphic to*

$$U_{2,4}, F_7, F_7^*, M(K_5)^*, \text{ or } M(K_{3,3})^*.$$

Here K_5 and $K_{3,3}$ are the Kuratowski graphs, the complete graph on five vertices, and the complete bipartite graph on two sets of three vertices, respectively. Also, $M(G)$ is the polygon or cycle matroid of the graph G, and $M(G)^*$ is the orthogonal matroid of $M(G)$, namely the bond matroid of G. By duality, a matroid is cographic if and only if it has no minor isomorphic to $U_{4,2}$, F_7, F_7^*, $M(K_5)$, or $M(K_{3,3})$. The excluded minor characterization of planar graphic matroids is a very pleasing generalization of Kuratowski's Theorem, which states that a graph is planar if and only if it has no homeomorphic image of a subgraph isomorphic to K_5 or $K_{3,3}$. A matroid is planar graphic if and only if it has no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M(K_5)$, $M(K_5)^*$, $M(K_{3,3})$, $M(K_{3,3})^*$, or, equivalently, if and only if it is graphic with no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. Thus the planar graphic matroids are precisely those matroids which are both graphic and cographic. One more interesting subclass of the graphic matroids is the class of series-parallel matroids, which are characterized by the excluded minors $U_{2,4}$ and $M(K_4)$.

A number of interesting relations may be deduced from these excluded minor characterizations. For example, a hereditary class is closed under duality if and only if the dual of each excluded minor is also an excluded minor. This is the case for each of the classes considered above, except graphic and cographic matroids, which are duals of each other.

We can also see that a hereditary class A is contained in another hereditary class A' if and only if every excluded minor of A' has itself some minor which is an excluded minor for A. For example, graphic and cographic matroids are unimodular, and unimodular matroids are binary as well as ternary.

We now turn to general necessary and sufficient conditions for coordinatization. The following result of Tutte was the first such set of conditions and it was also an important step in his proof of the excluded minor characterization of unimodular matroids.

1.5.5. Proposition. *Let $M(S)$ be a matroid and assume that for every hyperplane (or copoint) H of M is given a function $F_H: S \to K$, where K is a field, so that*

(1) *kernel $F_H = H$ for every hyperplane H.*

(2) *For every three hyperplanes H_1, H_2, H_3 of M containing a common coline, there exist constants $\alpha_1, \alpha_2, \alpha_3, \in K$, all non-zero, such that $\alpha_1 F_{H_1} + \alpha_2 F_{H_2} + \alpha_3 F_{H_3} = 0$.*

Then M may be coordinatized over K. Conversely, any coordinatization of M over K may be used to construct functions F_H satisfying (1) *and* (2).

In order to prove this proposition, we first need a lemma. Let W denote the vector space of all functions from S into K, and V the subspace of W spanned by $\{f_H | H \text{ is a hyperplane of } M\}$.

1.5.6. Lemma. *Let $\{f_H\}$ be given satisfying the hypotheses of Proposition 1.5.5, and let $B=\{b_1,b_2,\dots,b_n\}$ be a basis of $M(S)$. Then the functions f_{H_i} corresponding to the basic hyperplanes $H_i=\overline{B-b_i}$ form a basis of V.*

Proof of lemma. $A=\{f_{H_1},f_{H_2},\dots,f_{H_n}\}$ is linearly independent in V, for $f_{H_i}(b_j)\neq 0$ if and only if $i=j$, for $1\leqslant i\leqslant n, 1\leqslant j\leqslant n$. It remains to be shown that $f_H\in\text{span } A$ for every hyperplane H.

Let H be an arbitrary hyperplane of M, and let $h=n-1-|H\cap B|$. We use induction on h, noting that the case $h=0$ is trivial, since then $f_H\in A$.

Assume by induction hypothesis that $f_J\in\text{span } A$ for all hyperplanes J such that $n-1-|J\cap B|<h$.

Now we assume by re-indexing that $H\cap B=\{b_1,b_2,\dots,b_l\}, l=n-h-1$. Since $H\cap B$ is independent, we may extend it to a basis $\{b_1,b_2,\dots,b_l,a_{l+1},a_{l+2},\dots,a_{n-1}\}$ of H. Then

$$L=\overline{\{b_1,b_2,\dots,b_l,a_{l+1},a_{l+2},\dots,a_{n-2}\}}$$

is a coline of M contained in H. By choosing $b'\in B-L, b''\in B-H'$, we construct distinct hyperplanes $H'=\overline{L\cup b'}$ and $H''=\overline{L\cup b''}$. Furthermore, $|H'\cap B|=|H''\cap B|=l+1$, hence H' and H'' are distinct from H, and $f_{H'}$ and $f_{H''}$ are in span A. But by hypothesis (2) of the proposition, since H, H' and H'' are distinct hyperplanes containing L, $f_H\in\text{span}\{f_{H'},f_{H''}\}\subseteq\text{span } A$, completing the proof of the lemma. □

Proof of Proposition 1.5.5. For any $s\in S$, we define a linear functional L_s on V by $L_s(f)=f(s)\in K$ for all $f\in V$. Then the mapping $\sigma:S\to V^*, s\to L_s$ will coordinatize $M(S)$ if we can show that independent and dependent sets are preserved under σ (since V^*, the dual space of V, is a vector space over K). Clearly it suffices to consider maximal independent sets, or bases of M, and minimal dependent sets.

Let $\{b_1,b_2,\dots,b_n\}=B$ be any basis of $M(S)$. Then from the lemma we obtain the basis $\{f_{H_1},f_{H_2},\dots,f_{H_n}\}$ of V, where $f_{H_i}(b_j)\neq 0$ if and only if $i=j$. Thus $L_{b_j}(f_{H_i})\neq 0$ if and only if $i=j$, so $L_{b_1},L_{b_2},\dots,L_{b_n}$ are independent in V^*.

Now let $\{b_0,b_1,\dots,b_k\}$ be a minimal dependent set in $M, k\leqslant n$. Then the independent set $\{b_1,b_2,\dots,b_k\}$ may be extended to a basis $\{b_1,b_2,\dots,b_n\}=B$ of $M(S)$. As before, the lemma provides a basis $\{f_{H_1},f_{H_2},\dots,f_{H_n}\}$ of V with $L_{b_j}(f_{H_i})\neq 0$ if and only if $i=j$. But $b_0\in\overline{\{b_1,b_2,\dots,b_k\}}\subseteq\overline{B-\{b_i\}}$ for all $i>k$. Thus $L_{b_0}(f_{H_i})=0$ for all $i>k$. Since the linear functional L_{b_0} is determined by its values on the basis $\{f_{H_1},f_{H_2},\dots,f_{H_n}\}$ of V, we have $L_{b_0}=\sum_{i=1}^k\alpha_iL_{b_i}$, where $\alpha_i=L_{b_0}(f_{H_i})/L_{b_i}(f_{H_i})$. Thus $L_{b_0},L_{b_1},\dots,L_{b_k}$ are linearly dependent in V^*, completing the proof of the sufficiency of (1) and (2).

The converse is easy to prove. If $\zeta:S\to V$ is a coordinatization of M over K, then for any hyperplane $H, \zeta(H)$ spans a subspace U which is a hyperplane of V (that is, a subspace of dimension one less than V). Now, there is a unique (up to non-zero scalar multiple) linear functional $f_U: V \to K$ whose kernel is U, and $f_H = f_U \circ \zeta$ is the desired function, since conditions (1) and (2) may easily be checked. □

Another sufficient condition for coordinatization, due to Kantor (1975), is that each coline has at least three hyperplanes and each rank 4 minor is coordinatizable over a fixed prime field $GF(p)$.

1.6. Brackets

Among the most useful general conditions for coordinatizability are the bracket conditions. If $\zeta: M(S) \to V$ is a coordinatization into a vector space V of dimension n over a field K, where $n = \text{rank } M$, and if vectors in V are expressed as column vectors with respect to a standard basis B, then for any $x_1, x_2, \ldots, x_n \in S$, we define $[x_1, x_2, \ldots, x_n] = \det(\zeta x_1, \zeta x_2, \ldots, \zeta x_n)$. These determinants are called the brackets of ζ, and are often denoted $[X]$, where X is the sequence $(x_1, x_2, \ldots, x_n)$.

The following proposition is closely related to a result widely known to invariant theorists in the nineteenth century. This result says that assigning values to the brackets so that certain relations (called syzygies) are satisfied determines (uniquely, up to linear equivalence) a set of vectors having the assigned bracket values. Thus a map of S into V is determined simply by specifying the values of the brackets arbitrarily, provided the syzygies are satisfied. However, this classical result did not predetermine which bracket values were to be zero.

1.6.1 Proposition. *Let $M(S)$ be a matroid of rank n, and let $[x_1, x_2, \ldots, x_n]$ be assigned a value in the field K, for every $x_1, x_2, \ldots, x_n \in S$. A necessary and sufficient condition for the existence of a coordinatization ζ of M over K whose brackets are precisely the assigned values is that the following relations (or syzygies) be satisfied:*

(1) *$[x_1, x_2, \ldots, x_n] = 0$ if and only if $\{x_1, x_2, \ldots, x_n\}$ is either dependent in M or contains fewer than n distinct elements.*

(2) *(Antisymmetry) $[x_1, x_2, \ldots, x_n] - (sgn\,\sigma)[x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}] = 0$ for every permutation σ of $\{1, 2, \ldots, n\}$, for every $x_1, x_2, \ldots, x_n \in S$.*

(3) *$[x_1, x_2, \ldots, x_n]\ \ [y_1, y_2, \ldots, y_n] - \sum_{i=1}^{n} [y_i, x_2, \ldots, x_n]\ \ [y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n] = 0$ for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in S$.*

Proof: We first check the necessity. Let ζ be a coordinatization of $M(S)$. From elementary properties of determinants, we see immediately that (1) and (2) are satisfied by the brackets of ζ. To verify (3), we first note that the equation is trivial unless some summand is non-zero, and hence either $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are both bases of M, or else for some i, $\{y_i, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n\}$ are both bases. In fact, we may assume the former of these, for if $\{y_i, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n\}$ are both bases, the syzygy of type (3) with $[y_i, x_2, \ldots, x_n]$ $[y_1, y_2, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n]$ as first term is easily checked to be equivalent to the original syzygy with $[x_1, \ldots, x_n][y_1, \ldots, y_n]$ as first term, using antisymmetry. We now apply the non-singular linear transformation $T: V \to V$ which maps ζx_j to the j-th unit vector $\mathbf{e}_j$ of V, for each j. Let $T(\zeta y_j) = \mathbf{w}_j \in V$, and let W be the $n \times n$ matrix whose j-th column is $\mathbf{w}_j$. Applying T multiplies every determinant in (3) by the same constant, hence (3) is equivalent to

$$\begin{aligned}(\det I)(\det W) &= \sum_{i=1}^{n} \det(\mathbf{w}_i, \mathbf{e}_2, \ldots, \mathbf{e}_n) \det(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{i-1}, \mathbf{e}_1, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_n) \\ &= \sum_{i=1}^{n} w_{1i}(-1)^{i-1} \det W_{1i}, \end{aligned} \tag{1.1}$$

where W_{1i} is the minor of W with row 1 and column i deleted. But equation (1.1) is just the Laplace expansion of det W by its first row. Since T is invertible, the syzygy (3) is verified.

We now prove the sufficiency. We assume that $[x_1, x_2, \ldots, x_n]$ is given as an element of K for every $x_1, x_2, \ldots, x_n \in S$ so that the syzygies are satisfied. We must construct a coordinatization ζ whose brackets are equal to the assigned values, that is

$$\det(\zeta x_1, \zeta x_2, \ldots, \zeta x_n) = [x_1, x_2, \ldots, x_n]. \tag{1.2}$$

Let $Y = \{y_1, y_2, \ldots, y_n\}$ be a basis of $M(S)$. Then $[Y] \neq 0$, and we may normalize the bracket values by dividing each of them by $[Y]$. Since the syzygies are each homogeneous, they are still satisfied by the normalized bracket values, and thus we may assume $[Y] = 1$. We now define the i-th coordinate of the vector $\zeta(x)$ by $\zeta(x)^i = [y_1, y_2, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n]$. We will now show that $\zeta: S \to K^n$ is the desired coordinatization. Actually, it suffices to show that (1.2) holds for all $x_1, x_2, \ldots, x_n$, for then the fact that ζ is a coordinatization follows from syzygy (1).

Let $x_1, x_2, \ldots, x_n \in S$ be arbitrary. We may assume that these n elements are distinct, for otherwise $[x_1, x_2, \ldots, x_n] = \det(\zeta x_1, \zeta x_2, \ldots, \zeta x_n) = 0$. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $k = |X - Y|$. We now show (1.2) by induction on k. If $k = 0$ or 1, then (1.2) holds by the definition of ζ, so suppose $k \geqslant 2$. Then, using the induction hypothesis,

$$\begin{aligned}[X][Y] &= \sum_{i=1}^{n} [y_i, x_2, \ldots, x_n][y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_n] \\ &= \sum_{i=1}^{n} \det(\zeta y_i, \zeta x_2, \ldots, \zeta x_n) \det(\zeta y_1, \ldots, \zeta y_{i-1}, \zeta x_1, \zeta y_{i+1}, \ldots, \zeta y_n) \\ &= \det(\zeta X)\det(\zeta Y)\end{aligned}$$