Cellular structures in topology

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CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York Port Chester

Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1990

Printed in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data

Fritsch, Rudolf, *1939–*
Cellular structures in topology.
1. Topology
I. Title II. Piccinini, R. (Renzo), *1933–*
514

Library of Congress cataloging in publication data

Fritsch, Rudolf,
Cellular structures in topology / Rudolf Fritsch, Renzo A. Piccinini.
p. cm.–(Cambridge studies in advanced mathematics)
Bibliography: p.
Includes index.
ISBN 0 521 32784 9
1. CW complexes. 2. Complexes. 3. *k*-spaces. I. Piccinini,
Renzo A., 1933– . II. Series.
QA611.35.F75 1990
514'.223–dc20 89–35772 CIP

ISBN 0 521 32784 9

TM

Contents

1

The fundamental properties of CW-complexes

Balls and balloons are the standard models for the cells used in the theory of CW-complexes; thus, the chapter starts by 'playing' a bit with such toys. Next, it continues with a discussion of the problem of attaching n-cells to a space and with the actual construction of CW-complexes, followed by a detailed study of the fundamental properties of such spaces.

The unusual number given to the first section of this chapter, namely 1.0, stems from the fact that the material discussed therein is really very elementary.

1.0 Balls, spheres and projective spaces

The *ball* in the Euclidean space $\mathbf{R}^{n+1}$ is the space

$$B^{n+1} = \{s = (s_0, s_1, \ldots, s_n) : |s| \leqslant 1\};$$

its topological boundary is the *sphere*

$$\delta B^{n+1} = S^n = \{s \in B^{n+1} : |s| = 1\}$$

and the difference

$$\mathring{B}^{n+1} = B^{n+1} \backslash S^n$$

is the interior of the ball B^{n+1}, namely, the *open ball.* Observe that the ball $B^1 = [-1, 1]$ does not coincide with the unit interval $I = [0, 1]$ (in the sequel, the boundary of I will be denoted by $\dot{I}$).

Intuitively, one may view a sphere as the skin of a ball (i.e., a balloon). To blow up a balloon, there must be an opening, a 'base point'; thus, set the point $e_0 = (1, 0, \ldots, 0)$ as the base point of both B^{n+1} and S^n.

Spheres do not appear only as boundaries of balls; in addition to the inclusions

$$i^n : S^n \to B^{n+1},$$

it will be necessary to discuss several standard maps relating spheres and balls. The list of such maps described in this section is actually longer than that needed to develop the material herein. The primary two reasons are:

these maps could be used to fill in the details for the material sketched in the appendix;

some of the maps discussed could be used in the homology of cellular structures (e.g., the Hurewicz isomorphism theorem). Although homology is beyond the scope of this volume, it is a natural continuation for the theory here developed.

It is often convenient to view all balls B^{n+1} and all spheres S^n as contained in the space $\mathbf{R}^\infty$ of all sequences which vanish almost everywhere, via the embeddings $s \mapsto (s, 0, 0, \ldots)$; the topology of $\mathbf{R}^\infty$ is determined by the family of all Euclidean subspaces $\mathbf{R}^n$ (see Section A.2). Within this framework, consider the origin of $\mathbf{R}^\infty$ as the *0-ball*

$$B^0 = \{0\},$$

whose boundary is the 'sphere'

$$\delta B^0 = S^{-1} = \varnothing,$$

and which coincides with its interior

$$\mathring{B}^0 = B^0.$$

In contrast with these 'minimal' models B^0 and S^{-1}, one has the *infinite ball* $B^\infty = \bigcup_{n \geqslant 0} B^n$ and the *infinite sphere* $S^\infty = \bigcup_{n \geqslant 0} S^n$ as subspaces of R^∞. Notice that these two infinite models are determined by the corresponding families of finite models (see Corollary A.2.3).

The ball B^n is embedded into the ball B^{n+1} as a strong deformation retract; a suitable retraction is the map

$$j^n : B^{n+1} \to B^n,$$

given by

$$j^n(s) = (s_0, \ldots, s_{n-1}).$$

Define the 'eggs of Columbus' using the map j^n, i.e. the inclusions

$$j_+, j_- : B^{n+1} \rightarrowtail B^{n+1}$$

given by

$$j_+(s) = (s_0, \ldots, s_{n-1}, \tfrac{1}{2}(s_n + \sqrt{1 - |j^n(s)|^2}))$$

and

$$j_-(s) = (s_0, \ldots, s_{n-1}, \tfrac{1}{2}(s_n - \sqrt{1 - |j^n(s)|^2})).$$

The function j_+ (resp. j_-) maps the upper (resp. lower) hemisphere onto itself and the lower (resp. upper) hemisphere onto the equatorial ball B^n (see Figure 1).

The deformation

$$d^n : (B^n \times B^n) \times I \to B^n \times B^n$$

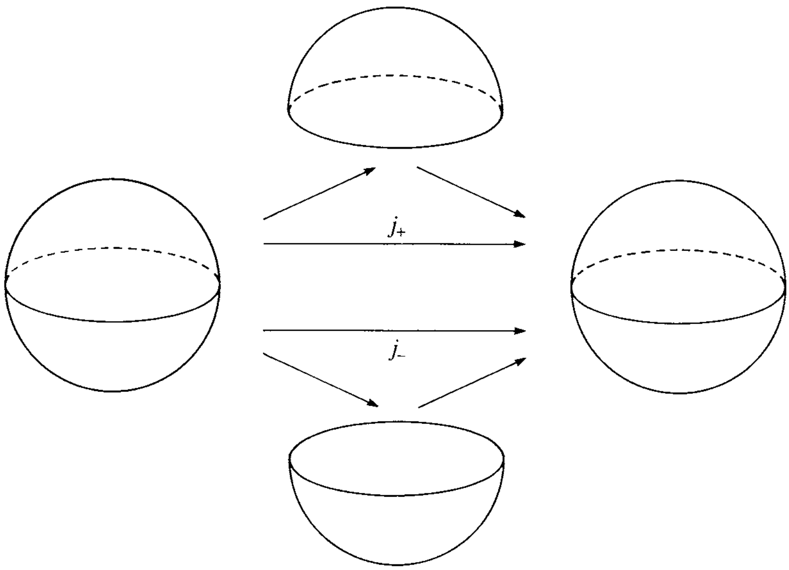

Figure 1

defined by

$$d^n((s,s'),t) = ((1-t)s + t(\tfrac{1}{2}(s+s')), (1-t)s' + t(\tfrac{1}{2}(s+s'))),$$

for every $(s,s') \in B^n \times B^n$ and every $t \in I$, shows that the diagonal subspace $\Delta B^n \subset B^n \times B^n$ is a strong deformation retract of $B^n \times B^n$; thus, balls are LEC spaces (see Section A.4, page 253).

The sphere S^{n-1} is included into the sphere S^n as its equator, and this inclusion, in turn, extends to embeddings

$$i_-, i_+ : B^n \rightarrowtail S^n$$

of the ball B^n into the southern, respectively northern hemisphere of S^n, given by

$$i_-(s) = (s, -\sqrt{1-|s|^2})$$

and

$$i_+(s) = (s, \sqrt{1-|s|^2}),$$

respectively, having $j^n|S^n$ as common left inverse.

The maps i_-, i_+ are homotopic only in a very curious way; in fact, a homotopy can be constructed by observing that both maps are homotopic to the constant map onto the base point, but there is no homotopy between them relative to the boundary (see the end of this paragraph). Viewed as maps into B^{n+1} the maps i_-, i_+ are homotopic in a neat manner namely, rel. S^{n-1} via the map

$$h^n : B^n \times I \to B^{n+1},$$

given by

$$h^n(s,t) = (s, (2t-1)\sqrt{1-|s|^2}).$$

The importance of this map h^n resides in the fact that every homotopy rel. S^{n-1} given between two maps defined on B^n factors through h^n. In particular this shows: If i_-, i_+ were homotopic rel. S^{n-1}, any corresponding homotopy factored through h^n would yield a retraction of B^{n+1} onto S^n, contradicting Brouwer theorem (see TheoremA.9.4).

Next, recall that the map (Figure 2)

$$c^n : S^n \times I \to B^{n+1}$$

given by

$$c^n(s,t) = (1-t)e_0 + ts$$

induces a homeomorphism

$$S^n \wedge I \to B^{n+1}$$

where the symbol $\wedge$ denotes the usual *smash product*

$$S^n \wedge I = S^n \times I / S^n \times \{0\} \cup \{e_0\} \times I.$$

The formation of the smash product with one factor equal to I is also known as the *reduced cone construction*. The *reduced suspension* of a based space (X, x_0) is one step further away; this construction is given on the based space (X, x_0) by

$$\Sigma.X = X \times I / X \times \dot{I} \cup x_0 \times I$$

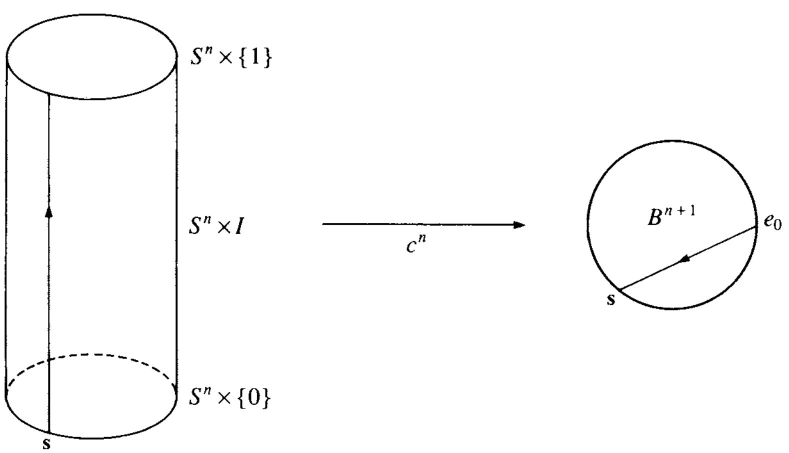

Figure 2

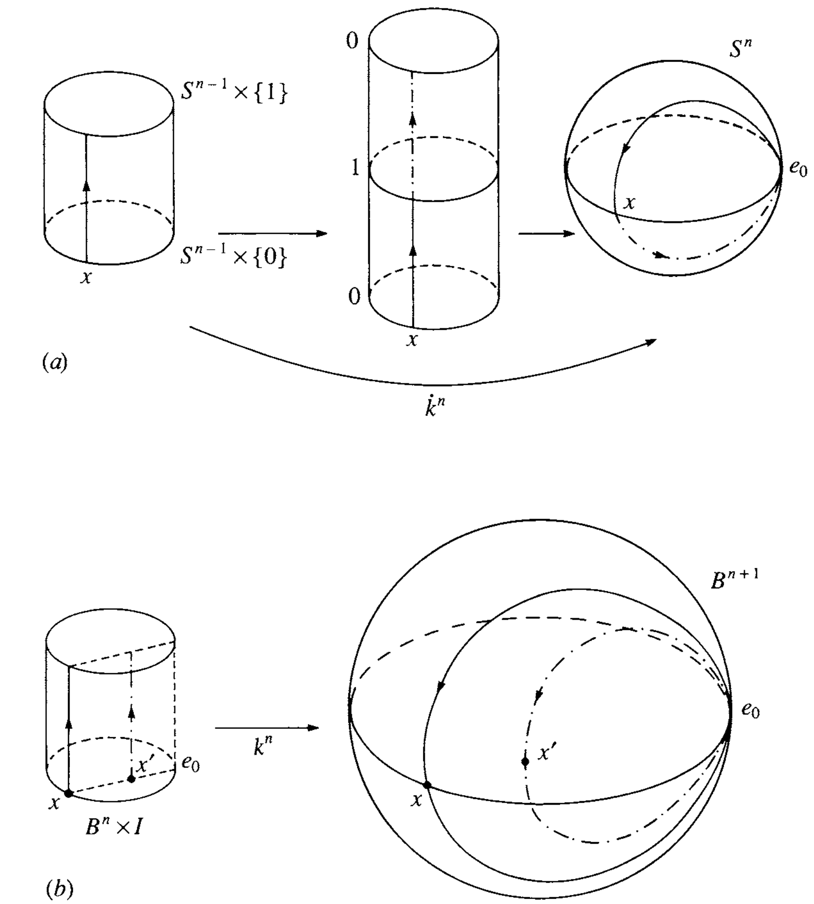

Figure 3

(note that $\Sigma.X$ is homeomorphic to the smash product $X\wedge S^1$); if $f:(Y,y_0)\to(X,x_0)$ is a based map, its suspension

$$\Sigma.f:\Sigma.Y\to\Sigma.X$$

is the map induced by $f\times 1 : Y\times I\to X\times I$.

For $n\geqslant 1$, define $\dot{k}^n : S^{n-1}\times I\to S^n$ (see Figure 3(*a*)) by

$$\dot{k}^n(s,t)=\begin{cases} i_+c^{n-1}(s,2t), & 0\leqslant t\leqslant \frac{1}{2}\\ i_-c^{n-1}(s,2-2t), & \frac{1}{2}\leqslant t\leqslant 1;\end{cases}$$

the map $\dot{k}^n$ takes $S^{n-1}\times\dot{I}\cup e_0\times I$ into e_0 and is bijective outside that space; thus, it induces a homeomorphism $\Sigma.S^{n+1}\to S^n$. Moreover, the map $\dot{k}^n$ can be extended to a map $k^n : B^n\times I\to B^{n+1}$ (see Figure 3(*b*))

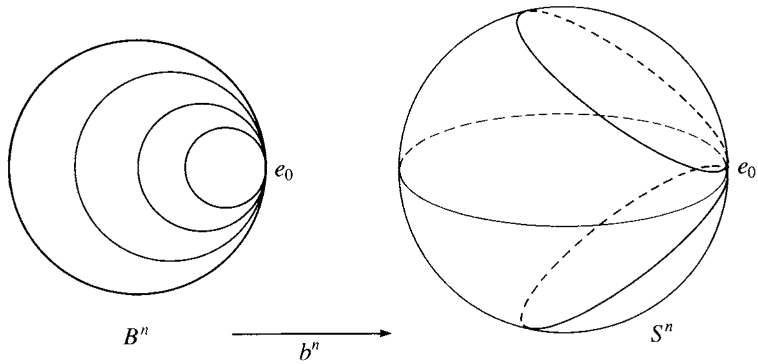

Figure 4

simply by taking

$$k^n(c^{n-1}(s,t'),t) = c^n(\dot{k}^n(s,t),t');$$

this latter map induces a homeomorphism $\Sigma.B^n \to B^{n+1}$. Finally, notice that the map $\dot{k}^n$ factors through the map c^{n-1}, and thus induces a map

$$b^n : B^n \to S^n;$$

formally, $b^n \circ c^{n-1} = \dot{k}^n$. In turn, the map b^n gives a homeomorphism between B^n/S^{n-1} and S^n. It is convenient to extend the definition of b^n to include $b^0 : B^0 \to S^0$ given by $b^0(B^0) = \{-1\}$. Figure 4 indicates that b^n is homotopic rel. $\{e_0\}$ to i_+ via a homotopy moving S^{n-1} only in the lower hemisphere.

The following maps are relevant to the definition of homotopy groups:

(i) the *units*

$$u^n : B^{n+1} \to B^{n+1}, \qquad \dot{u}^n : S^n \to S^n$$

defined for all $n \in \mathbf{N}$ as the constant-based maps;

(ii) the *inversions*

$$l^n : B^{n+1} \to B^{n+1},$$

defined by $l^n(k^n(s,t)) = k^n(s,1-t)$ for every $(s,t) \in B^n \times I$; this inversion on B^{n+1} induces an inversion $\dot{l}^n : S^n \to S^n$ on S^n; notice that l^n, $\dot{l}^n$ are reflections about the hyperplane $\mathbf{R}^n \subset \mathbf{R}^{n+1}$:

$$l^n(s_0,\ldots,s_n,s_{n+1}) = (s_0,\ldots,s_n,-s_{n+1});$$

(iii) for $n \geqslant 1$, the *pinchings* (see Figure 5)

$$p^n : B^{n+1} \to B^{n+1} \vee B^{n+1}$$

given by

$$p^n(k^n(s,t)) = \begin{cases} (k^n(s,2t),e_0), & 0 \leqslant t \leqslant \frac{1}{2} \\ (e_0,k^n(s,2t-1)), & \frac{1}{2} \leqslant t \leqslant 1; \end{cases}$$

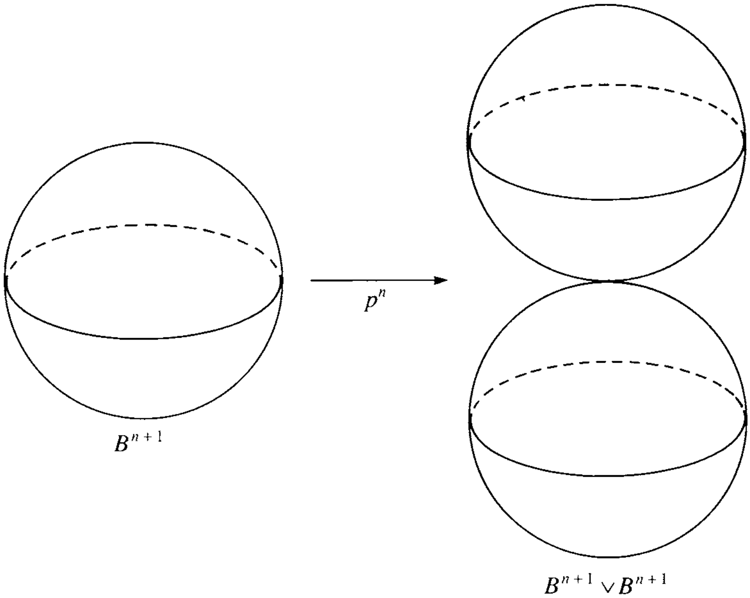

Figure 5

this means that the points with last coordinate equal to zero are mapped into the wedge point (e_0, e_0).

The maps p^n induce the *pinching of the spheres*

$$\dot{p}^n : S^n \to S^n \vee S^n.$$

(The symbol $\vee$ denotes the usual *wedge product*: for any pair of based spaces, say (X, x_0), (Y, y_0), the space $X \vee Y$ is defined to be $X \times \{y_0\} \cup \{x_0\} \times Y$, regarded as a subspace of $X \times Y$.)

An inaccurate but graphic description of the pinching is provided by cell division, a basic process in biology.

For $n \geqslant 2$, there is another useful type of *pinching*:

$$\hat{p} : B^{n+1} \to B^{n+1} \vee B^{n+1}$$

given by

$$\hat{p}^n(k^n(k^{n-1}(s,u),t)) = \begin{cases} (k^n(k^{n-1}(s,2u),t), e_0), & 0 \leqslant u < \frac{1}{2}, \\ (e_0, k^n(k^{n-1}(s,2u-1),t)), & \frac{1}{2} \leqslant u \leqslant 1. \end{cases}$$

This means that the points with penultimate coordinate equal to zero are mapped into the wedge point (e_0, e_0).

Next, consider the map obtained by projecting $B^{n+1} \times I$ onto $B^{n+1} \times \{0\} \cup S^n \times I$ from $(0, 2)$ in $\mathbf{R}^{n+1} \times \mathbf{R}$ (see Figure 6):

$$r^{n+1} : B^{n+1} \times I \to B^{n+1} \times \{0\} \cup S^n \times I$$

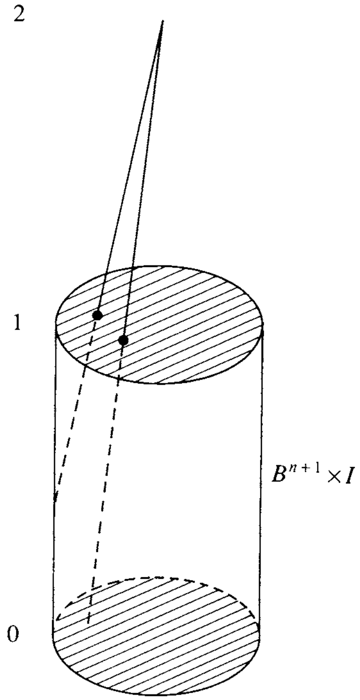

Figure 6

$$r^{n+1}(s,t) = \begin{cases} \dfrac{2}{2-t}(s,0), & 0 \leqslant t \leqslant 2(1-|s|), \\ \dfrac{1}{|s|}(s, 2|s|+t-2), & 2(1-|s|) \leqslant t \leqslant 1, |s| \neq 0. \end{cases}$$

Notice that the restriction of r^{n+1} to $B^{n+1} \times \{0\} \cup S^n \times I$ is the identity and that the composition of r^{n+1} with the inclusion of the latter space into $B^{n+1} \times I$ is homotopic rel. $B^{n+1} \times \{0\} \cup S^n \times I$ to the identity map, via the homotopy

$$R^{n+1} : B^{n+1} \times I \times I \to B^{n+1} \times I$$

given by

$$R^{n+1}(s,t,u) = u(s,t) + (1-u)r^{n+1}(s,t);$$

thus, $B^{n+1} \times \{0\} \cup S^n \times I$ is a strong deformation retract of $B^{n+1} \times I$. This means that the inclusion of S^n in B^{n+1} is a closed cofibration (see Example 1, Section A.4).

The restriction of the homotopy R^n to $B^n \times \{1\} \times I \simeq B^n \times I$ factors through the map h^n, thereby inducing a homeomorphism (see Figure 7)

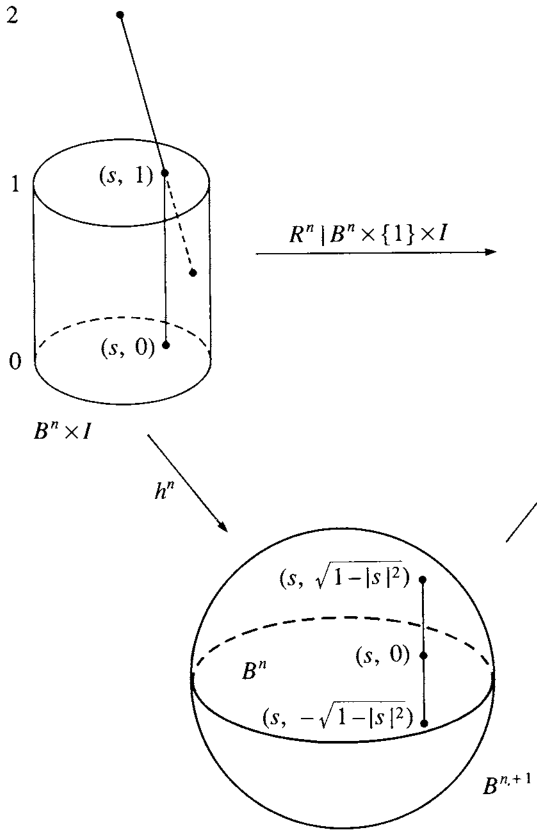

Figure 7

$$v^n : B^{n+1} \to B^n \times I;$$

one should notice that, regarding i_+, i_- as inclusions of B^n into B^{n+1}, then

$$v^n \circ i_- = r^n | B^n \times \{1\} \quad \text{and} \quad v^n \circ i_+ = \text{inclusion}.$$

The homeomorphism v^n, interesting in its own right, can be used to interchange the components $B^n \times \{0\} \cup S^{n-1} \times I$ and $B^n \times \{1\}$ of the boundary of $B^n \times I$: to see this, first note that v^n maps the upper hemisphere of S^{n+1} onto $B^n \times \{1\}$ and its lower hemisphere onto $B^n \times \{0\} \cup S^{n-1} \times I$; the actual interchange is then effected by the composite function $w^n = v^n \circ l^n \circ (v^n)^{-1}$. Two more remarks about the map v^n are called for: firstly, v^n induces a homeomorphism

$$\dot{v}^n : S^n \to B^n \times \dot{I} \cup S^{n-1} \times I;$$

secondly, v^n combines with the two pinchings p^n and $\hat{p}^n$ to yield an interesting commutative property:

Lemma 1.0.1 *For every $n \geq 1$,*

(i) *there is a unique map*

$$q^n : (B^n \vee B^n) \times I \to (B^n \times I) \vee (B^n \times I)$$

such that

$$q^n \circ (p^n \times 1) \circ v^n = (v^n \vee v^n) \circ \hat{p}^n;$$

(ii) *the map*

$$\hat{q}^n : (B^n \times I) \vee (B^n \times I) \to (B^n \vee B^n) \times I$$

induced by the obvious inclusions is a left homotopy inverse to q^n: there is a homotopy $\hat{q}^n \circ q^n \simeq 1$ rel. $((e_0, e_0), 1)$ and transforms the boundary of $(B^n \vee B^n) \times I$ into itself. □

In order to have enough fun in this game of balls and balloons, one actually needs more than one ball and one balloon in every dimension. Thus every space homeomorphic to the ball B^n (respectively, $\mathring{B}^n$) is called an *n-ball* (respectively, *open n-ball*) and every space homeomorphic to the sphere S^n is called an *n-sphere*. If B is any $(n+1)$-ball, its boundary sphere i.e., the image of S^n under a homeomorphism $B^{n+1} \to B$, is denoted by δB.

Proposition 1.0.2 *For any non-negative integers p and q, $B^p \times B^q$ is a $(p+q)$-ball with boundary sphere $B^p \times S^{q-1} \cup S^{p-1} \times B^q$; moreover, for every $n > 0$, $(B^1)^n$ is an n-ball.*

Proof Define $\Phi : B^p \times B^q \to B^{p+q}$ by setting, for every $(s, s') \in B^p \times B^q$,

$$\Phi(s, s') = \{\max(|s|, |s'|)/\sqrt{|s|^2 + |s'|^2}\}(s, s'),$$

if $(s, s') \neq (0, 0)$ and

$$\Phi(0, 0) = 0.$$

The continuity of Φ is not difficult to prove. Its inverse is obtained as follows. Let $s = (s_1, \ldots, s_p, \ldots, s_{p+q}) \in B^{p+q}$ be given. Set $s' = (s_1, \ldots, s_p)$ and $s'' = (s_{p+1}, \ldots, s_{p+q})$; then, define

$$\Phi^{-1}(s) = \{|s|/\max(|s'|, |s''|)\}(s', s'').$$

The restriction of Φ to $\delta(B^p \times B^q)$ gives the second homeomorphism announced in the statement. The third homeomorphism is obtained by induction on n. □

Projective spaces

From the topological point of view, projective spaces are intimately connected to spheres. However, before exhibiting this connection, one must give the definition of 'projective space' over a field.

Let $\mathbf{F}$ be a (not necessarily commutative) field. The *n-dimensional projective space over* $\mathbf{F}$, denoted by $\mathbf{F}P^n$, is defined as the set of all 1-dimensional (left) vector subspaces of the $(n+1)$-dimensional (left) vector space $\mathbf{F}^{n+1}$. The space $\mathbf{F}P^n$ can be identified with the set $(\mathbf{F}^{n+1}\setminus\{0\})/\sim$, where $\sim$ is the equivalence relation defined by: $s \sim s'$ iff there is a scalar $t \in \mathbf{F}$ with $s' = ts$.

If $\mathbf{F}$ is a topological field the projective space $\mathbf{F}P^n$ is given the identification topology induced by the projection $\mathbf{F}^{n+1}\setminus\{0\} \to \mathbf{F}P^n$. In this book, $\mathbf{F}$ represents the field $\mathbf{R}$ of real numbers, the field $\mathbf{C}$ of complex numbers or the skew-field $\mathbf{H}$ of quaternions. Then the space $\mathbf{F}^{n+1}$ can be identified with one of the Euclidean spaces $\mathbf{R}^{n+1}$, $\mathbf{R}^{2n+2}$ or $\mathbf{R}^{4n+4}$. Note that one can find, for every point in the projective space, a representative of length 1 in the corresponding Euclidean space, i.e., a point in the spheres S^n, S^{2n+1} or S^{4n+3}. These identifications yield, respectively, the identification maps

$$q_{\mathbf{R}}{}^n : S^n \to \mathbf{R}P^n,$$
$$q_{\mathbf{C}}{}^n : S^{2n+1} \to \mathbf{C}P^n,$$
$$q_{\mathbf{H}}{}^n : S^{4n+3} \to \mathbf{H}P^n.$$

The inverse image of a point in the projective space is a pair of antipodal points in S^n for $\mathbf{F} = \mathbf{R}$, a circle ($=$ 1-sphere) in S^{2n+1} for $\mathbf{F} = \mathbf{C}$ and a 3-sphere in S^{4n+3} for $\mathbf{F} = \mathbf{H}$.

1.1 Adjunction of *n*-cells

The reader should always bear in mind that all the work in this book is done within the context of the category of weak Hausdorff k-spaces, denoted simply by *Top* (except in Section A.1, where it is denoted by *wHk*(*Top*)).

Intuitively, a CW-complex is a space which can be considered as a union of disjoint 'open cells'. For instance, the ball B^{n+1} can be considered as the union of an $(n+1)$-cell, namely the open ball $\mathring{B}^{n+1}$, an n-cell, namely the punctured sphere $S^n\setminus\{e_0\}$, and the 0-cell $\{e_0\}$:

$$B^{n+1} = \mathring{B}^{n+1} \cup (S^n\setminus\{e_0\}) \cup \{e_0\}.$$

In this book the term 'cell' will often be preceded by the adjectives 'open', 'closed', 'regular', or the combination 'closed regular'. The following list is intended to make matters clear. A subspace e of a space X is said to be

an *open n-cell* in X ($n \in \mathbf{N}$), if it is an open n-ball (recall that an open n-ball is a space homeomorphic to the open ball $\mathring{B}^n$);
a *closed n-cell* in X, if it is the closure (in X) of an open n-cell;
a *regular n-cell* in X, if it is an open n-cell whose closure is an n-ball and whose boundary in the closure is an $(n-1)$-sphere;
a *closed regular n-cell* in X, if it is the closure of a regular n-cell.

Observe that an n-cell does not have to be regular: the punctured sphere $S^n \setminus \{e_0\}, n > 0$, as a subspace of the sphere S^n, is an example of this fact. For open, regular or closed regular n-cells e, the natural number n is the dimension of e: $\dim e = n$ (see Section A.9). By abuse of language, one also assigns to a closed n-cell the dimension n, although, outside the theory of CW-complexes, this does not necessarily coincide with the covering dimension of the space under consideration (see Example 5). But, if a space X contains an n-cell of any type, then $\dim X \geqslant n$, because inside each open n-cell there are closed n-balls (see Corollary A.9.2).

The ball B^{n+1} was decomposed into a union of open cells at the beginning of this section. In what follows, one should have this sort of cellular decomposition in mind. For the sake of simplicity, the formal constructions and proofs will often proceed in a slightly different manner.

A pair (X, A) is an *adjunction of n-cells*, $n \in \mathbb{N}$, if X can be viewed as an adjunction space (see Section A.4)

$$X = A \sqcup_f Y$$

where Y is a topological sum of n-balls and the domain of f consists of the boundary spheres of the balls forming Y; in other words, if X is given by a pushout of the form

$$\begin{array}{ccc} \bigsqcup_\lambda B_\lambda = Y & \longrightarrow & X \\ \uparrow & & \uparrow \\ \bigsqcup_\lambda S_\lambda & \longrightarrow & A, \end{array}$$

with B_λ an n-ball and $S_\lambda = \delta B_\lambda$, for all indices λ in an arbitrary index set Λ. If $n = 0$ the definition means simply that X is a topological sum of A and a discrete space. If (X, A) is an adjunction of n-cells, any path-component of $X \setminus A$ is an open n-cell in X, called an *n-cell* of (X, A). Each induced map $B_\lambda \to X$ is called a *characteristic map* for the λth cell; each induced map $S_\lambda \to A$ is an *attaching map* for the λth cell. If A is a based space and every map $S_\lambda \to A$ is based, the pair (X, A) is said to be a *based* adjunction of n-cells.

Proposition 1.1.1 *If (A, a_0) is path-connected and (X, A) is an adjunction of n-cells, $n > 0$, there exists a based adjunction of n-cells (X', A), such that X' is homotopically equivalent to X via homotopies rel. A.*

Proof Suppose that $X = A \sqcup_f (\sqcup_\lambda B_\lambda)$. Let $f_\lambda : S_\lambda \to A$ be the attaching map for the λth cell and let $\omega_\lambda : I \to A$ be a path such that $\omega_\lambda(0) = f_\lambda(e_0)$,

$\omega_\lambda(1) = a_0$; choose a representative for $(\omega_\lambda)_n^{-1}([f_\lambda])$ (see page 287 in the appendix), for every index λ. The maps f'_λ together define a based adjunction of n-cells (X', A) with the properties required (see Proposition A.4.15).

□

Example 1 For every $n > 0$, the pair (B^n, S^{n-1}) is an adjunction of just one regular n-cell; one can take the identity of S^{n-1} as an attaching map and the identity of B^n as a characteristic map.

Example 2 For every $n \in \mathbf{N}$, the pair $(S^n, \{e_0\})$ is an adjunction of just one non-regular n-cell. If $n > 0$, the map $b^n : B^n \to S^n$ (see page 6) can be used as a characteristic map; here there is no choice for the attaching map: it has to be the constant map.

Example 3 For every $n > 0$, the pair (S^n, S^{n-1}) is an adjunction of two regular n-cells. Take as components of the characteristic map the embeddings i_+, i_- (see page 3) of the ball B^n as the upper, respectively the lower, hemisphere into the sphere S^n.

The next example is not so trivial.

Example 4 For every $n \in \mathbf{N}$, the pair $(B^{n+1} \cup S^{n+1}, B^n \cup S^n)$ is an adjunction of exactly four regular $(n+1)$-cells. To prove this assertion, first observe that

$$B^{n+1} \cup S^{n+1} = B^{n+1} \sqcup_{B^n \cup S^n} (B^n \cup S^{n+1});$$

then note that because of the addition law (L3), it is enough to show that each of the pairs $(B^{n+1}, B^n \cup S^n)$ and $(B^n \cup S^{n+1}, B^n \cup S^n)$ is an adjunction of just two $(n+1)$-cells. Example 3 and the horizontal composition law (L1) are used to show that the pair $(B^n \cup S^{n+1}, B^n \cup S^n)$ is an adjunction of two $(n+1)$-cells. To prove that the pair $(B^{n+1}, B^n \cup S^n)$ is an adjunction of just two $(n+1)$-cells, construct the appropriate pushout using the 'eggs of Columbus' (see page 2) as components of the characteristic map.

Example 5 Let $f : B^1 \to B^n, n > 2$, be a *Peano curve*, i.e., a map from B^1 *onto* B^n. Then, the composition $f \circ j^1 | S^1$ defines a partial map $g : B^2 \dashrightarrow B^n$ (for the definition of the map j^1, see page 2). The pair $(B^n \sqcup_g B^2, B^n)$ is an adjunction of just one 2-cell. The corresponding closed 2-cell has covering dimension $n > 2$!

Example 6 Let $\mathbf{F}$ be one of the fields $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, of the real, complex or quaternionic numbers, respectively; also, let d be the dimension of $\mathbf{F}$ as a vector space over $\mathbf{R}$. Then, for every $n>0$, the pair $(\mathbf{F}P^n, \mathbf{F}P^{n-1})$ is an adjunction of just one non-regular dn-cell. The composition of the inclusion $i_+: B^{dn} \rightarrowtail S^{dn}$ (see page 3), the embedding $S^{dn} \rightarrowtail S^{dn+d-1}$, and the projection $q^n_{\mathbf{F}}: S^{dn+d-1} \to \mathbf{F}P^n$ (see page 11) may serve as characteristic map for the adjunction; this characteristic map induces the attaching map $q^{n-1}_{\mathbf{F}} : S^{dn-1} \to \mathbf{F}P^{n-1}$.

Proposition 1.1.2 *Let (X, A) be an adjunction of n-cells, say*

$$X = A \bigsqcup_f \left(\bigsqcup_{\lambda} B_\lambda \right).$$

Then the following statements hold true:

(i) *the inclusion $A \rightarrowtail X$ is a closed cofibration;*

(ii) *the space X is (perfectly) normal, whenever the subspace A* is *(perfectly) normal;*

(iii) *the space X has dimension n, whenever the subspace A is a normal space of dimension $\leqslant n$ and the index set is not empty;*

(iv) *$X \backslash A$ is a topological sum of open n-cells, one for each index λ;*

(v) *for any map $f': A \to A'$, the pair $(A' \bigsqcup_{f'} X, A')$ is an adjunction of n-cells.*

Proof The inclusion $dom\, f \rightarrowtail Y$ is a topological sum of closed cofibrations, and therefore is itself a closed cofibration. Thus (i) follows because the attaching process preserves cofibrations.

Since $\bigsqcup_\lambda B_\lambda$ is perfectly normal, the adjunction space X is (perfectly) normal if A is (perfectly) normal (see Proposition A.4.8 (iv)).

To prove (iii) note that under the first part of the condition given, the space X has dimension $\leqslant n$ (see Proposition A.4.8 (v)); if n-cells are really present, $\dim X \geqslant n$.

Part (iv) follows from the fact that $X \backslash A$ is homeomorphic to $Y \backslash dom\, f = \sqcup (B_\lambda \backslash S_\lambda)$.

Finally, (v) follows from the law of horizontal composition of Section A.4. □

Remark According to (iv), the index set for $\sqcup B_\lambda$ can be viewed as the set $\pi(X \backslash A)$ of path-components of $X \backslash A$. Give the discrete topology to the set $\pi(X \backslash A)$; then the space $B^n \times \pi(X \backslash A)$ can be viewed as the domain of the characteristic map for the adjunction of n-cells (X, A), and the space $S^{n-1} \times \pi(X \backslash A)$ can be viewed as the domain of the attaching map for the same adjunction. □

The bridge between the point of view of considering globally all the cells used in the adjunction, and that of considering successive attachings of single n-cells, is given by the following result.

Proposition 1.1.3 *The pair (X, A) is an adjunction of n-cells, iff*

(i) *for every path-component e of $X\backslash A$ the pair $(A \cup e, A)$ is an adjunction of just one n-cell ($A \cup e$ is considered as a subspace of X) and*

(ii) *the space X is determined by the family $\{A\} \cup \{\bar{e} : e \in \pi(X\backslash A)\}$.*

Proof '$\Rightarrow$': (i) Let e be an n-cell of (X, A) with attaching map f_e and characteristic map $\bar{f}_e$.

To prove the equality

$$A \cup e = A \bigsqcup\nolimits_{f_e} B_e,$$

observe first that $A \cup e = A \cup (B_e \backslash S_e)$, as sets. It remains to show that the subspace topology of $A \cup e$ is the same topology as that of the adjunction space $A \bigsqcup_{f_e} B_e$. Notice that by the universal property of the adjunction, the space $A \bigsqcup_{f_e} B_e$ has a finer topology than $A \cup e$. Next, let $V \subset A \cup e$ be such that $V \cap A$ is closed in A, and $\bar{f}_e^{-1}(V)$ is closed in B_e. Because X is a weak Hausdorff k-space, $V \cap \bar{f}_e(B_e) = \bar{f}_e(\bar{f}_e^{-1}(V))$ is closed in X (see Lemma A.1.1), and, hence, in $A \cup e$; this, together with the fact that $V \cap A$ is closed in A, implies that V is closed in X.

(ii) Let $U \subset X$ be such that $U \cap A$ and $U \cap \bar{e}$ are closed respectively in A and $\bar{e}$, for each $e \in \pi(X\backslash A)$. Then, if $\bar{f}$ is the characteristic map of the adjunction, $\bar{f}^{-1}(U) = \sqcup \bar{f}_e^{-1}(U \cap \bar{e})$ is closed.

'$\Leftarrow$': For every $e \in \pi(X\backslash A)$, let $f_e : S_e \to A$ denote an attaching map generating the adjunction space $A \cup e = A \bigsqcup_{f_e} B_e$. Let $f : \square S_e \to A$ be the map defined by the maps f_e, and let $\hat{X}$ be the adjunction space $A \bigsqcup_f (\bigsqcup B_e)$ with a fixed characteristic map $\bar{f}$. The pair $(\hat{X}, A)$ is an adjunction of n-cells, and thus it suffices to show that the spaces X and $\hat{X}$ coincide (up to canonical homeomorphism). The universal property of the adjunction space $\hat{X}$ gives rise to a bijective map $\hat{X} \to X$; thus, assume that the spaces $\hat{X}$ and X have the same underlying sets, and the topology of $\hat{X}$ is finer than that of X.

Notice that $\bar{f}(B_e) = \bar{e}$ because X is a weak Hausdorff k-space. Let $V \subset X$ be such that $V \cap A$ and $\bar{f}^{-1}(V)$ are closed in A and $\bigsqcup B_e$, respectively. Hence $\bar{f}^{-1}(V) \cap B_e$ is closed in B_e, for every $e \in \pi(X\backslash A)$; because $\bar{f}(\bar{f}^{-1}(V) \cap B_e) = V \cap \bar{e}$, it follows that $V \cap \bar{e}$ is closed in $\bar{e}$, for every $e \in \pi(X\backslash A)$. Condition (ii) implies that the set V is also closed in X. $\square$

The following result, which is actually contained in the previous proof, has some interest in its own right.

Lemma 1.1.4 *Let (X, A) be an adjunction of n-cells and let e be an n-cell of (X, A). Then,*

$$\bar{e} = \bar{f}(B),$$

where $\bar{f}$ denotes a characteristic map for e and B denotes an n-ball in the domain of $\bar{f}$. □

An advantage of looking at the adjunction of just one cell at a time lies in the fact that this process can be characterized without the explicit construction of a pushout diagram.

Lemma 1.1.5 *The pair (X, A) is an adjunction of just one n-cell iff*

(i) *A is closed in X*

and

(ii) *there is a map $B^n \to X$ inducing a homeomorphism $\mathring{B}^n \to X \setminus A$.*

Proof '⇒': clear from the definition.

'⇐': Let $\bar{f} : B^n \to X$ be a map as described in condition (ii). First, prove that $\bar{f}$ takes the boundary S^{n-1} of the ball B^n into the space A. To this end, assume the existence of a point $s \in S^{n-1}$ such that $\bar{f}(s) \in X \setminus A$. Then there is a unique point $s' \in \mathring{B}^n$ such that $\bar{f}(s) = \bar{f}(s')$; furthermore, the inverse image of every neighbourhood of $\bar{f}(s')$ contains points close to s, contradicting the assumption that $\bar{f}$ induces a homeomorphism $\mathring{B}^n \to X \setminus A$.

Denote by $f : S^{n-1} \to A$ the map induced by $\bar{f}$, and form the commutative square

$$\begin{tikzcd} B_n \arrow[r, "\bar{f}"] & X \\ S^{n-1} \arrow[u] \arrow[r, "f"'] & A \arrow[u] \end{tikzcd}$$

It remains to prove that X has the final topology with respect to $\bar{f}$ and the inclusion $A \subset X$. To show this, first observe that the subspace A is closed in X, by (i), and the subspace $\bar{f}(B^n)$ is closed in X, because X is weak Hausdorff. Since the space X is the union of these two closed subspaces, a function with domain X is continuous iff its restrictions to the subspaces A and $\bar{f}(B^n)$ are continuous. □

The condition (i) in this lemma is necessary, as one can deduce from the following.

Example 7 The pair $(B^2, B^2 \setminus \mathring{B}^1)$ satisfies condition (ii) for $n = 1$, but fails to be an adjunction of a 1-cell.

Because a space with a finite closed covering is determined by that covering, condition (ii) in Proposition 1.1.3 is superfluous if one deals with adjunctions of only finitely many cells. The following is an example showing that this condition is unavoidable in the general case.

Example 8 Let $\{B_\lambda : \lambda \in \mathbf{N}\backslash\{0\}\}$ be a countable set of copies of the ball B^1. For every index λ, let $\bar{f}_\lambda : B_\lambda \to I$ denote an embedding whose image is the interval $[1/(\lambda+1), 1/\lambda]$ and define $\bar{f} : \sqcup B_\lambda \to I$ by taking $\bar{f}|B_\lambda = \bar{f}_\lambda$. Since $\bar{f}(\sqcup S_\lambda)$ is contained in

$$A = \{0\} \cup \left\{\frac{1}{\lambda} : \lambda \in \mathbf{N}\backslash\{0\}\right\},$$

$\bar{f}$ induces a map $f : \sqcup S_\lambda \to A$ and a commutative square

$$\begin{tikzcd} \bigsqcup B_\lambda \arrow[r, "\bar{f}"] & I \\ \bigsqcup S_\lambda \arrow[u] \arrow[r, "f"'] & A \arrow[u] \end{tikzcd}$$

Now, for every index λ, define the 1-cell $e_\lambda = \bar{f}(\mathring{B}_\lambda)$ in I; then

$$A \cup e_\lambda = A \cup \left\{t \in \mathbf{R} : \frac{1}{\lambda+1} \leqslant t \leqslant \frac{1}{\lambda}\right\}$$

and the pair $(A \cup e_\lambda, A)$ is an adjunction of just one 1-cell (see Lemma 1.1.5). But I is not determined by the family $\{A \cup e_\lambda\}$! To see this, consider the sequence $\{(2\lambda+1)/2\lambda(\lambda+1)\}$. This sequence meets every space $A \cup e_\lambda$ in just one point, thus it is closed in the topology determined by $\{A \cup e_\lambda\}$; however, it converges to 0 in the usual topology of the unit interval I. (This situation may also serve as a counterexample in general topology: it is easy to see that, with respect to the topology of I determined by $\{A \cup e_\lambda\}$, 0 is a cluster point of $\bar{f}(\sqcup \mathring{B}_\lambda)$, but no sequence in $\bar{f}(\sqcup \mathring{B}_\lambda)$ converges to 0; this means that the resulting space is not a Fréchet space. Similar ideas will be used in Example 13 of the next section.)

There are two more relevant examples of pairs which are adjunctions of infinitely many n-cells.

Example 9 The concept of the wedge of two spheres $S^n \vee S^n$ was briefly discussed in Section 1.0; this concept has the following generalization. Let Γ be any set; for every $\gamma \in \Gamma$ take a copy of the n-sphere S^n with its base point e_0, i.e., $(S^n{}_\gamma, e_0) = (S^n, e_0)$. The *wedge product* of the family of based spaces

$$\{(S^n_\gamma, e_0) : \gamma \in \Gamma\}$$

(also called a *bouquet of n-spheres*) is the based space $(\vee_\Gamma S^n_\gamma, *)$, given by the set

$$\bigvee_\Gamma S^n_\gamma = \left\{ (s_\gamma) \in \prod_{\gamma\in\Gamma} S^n_\gamma : s_\gamma \neq e_0, \text{ for at most one } \gamma \in \Gamma \right\},$$

endowed with the final topology with respect to the canonical map

$$p : \bigsqcup_{\gamma\in\Gamma} S^n_\gamma \to \bigvee_{\gamma\in\Gamma} S^n_\gamma,$$

and the point $*$ taken to be the element (e_0). Note that if Γ is finite, this topology coincides with the subspace topology induced by $\prod_{\gamma\in\Gamma} S^n_\gamma$.

The pair $(\vee_\Gamma S^n, *)$ is an adjunction of n-cells; notice that there are as many n-cells as there are elements in Γ. A characteristic map for this adjunction is given by the map

$$\bar{f} : \bigsqcup_{\gamma\in\Gamma} B^n \cong B^n \times \Gamma \to \bigvee_\Gamma S^n$$

(here Γ is given the discrete topology) defined by $\bar{f}(s,\gamma) = (s_\gamma)$, where $s_\gamma = b^n(s)$.

Example 10 Let π be an abelian group and let n be a natural number > 1. Let $FA(\pi)$ be the free abelian group generated by the elements of π, and let Γ be a basis of the kernel of the canonical homomorphism $FA(\pi) \to \pi$. Let $\varphi : FA(\pi) \to \pi_n(\vee_\pi S^n, *)$ denote the homomorphism which assigns to a generator α of $FA(\pi)$ the homotopy class of the inclusion of S^n into the π-fold wedge $\vee_\pi S^n$ of S^n as the αth factor. Next, for each $\gamma \in \Gamma$, choose a representative $f_\gamma : S^n \to \vee_\pi S^n$ of the homotopy class $\varphi(\gamma)$. The maps f_γ define a partial map $f : B^{n+1} \times \Gamma \not\to \vee_\pi S^n$, whose resulting adjunction space $M(\pi, n)$ is called a *Moore space* of type (π, n). The construction of $M(\pi, n)$ shows that the pair $(M(\pi, n), \vee_\pi S^n)$ is an adjunction of $(n+1)$-cells.

What follows is more than just an example.

Theorem 1.1.6 (i) *Let (X, A) be an adjunction of n-cells and let $p : \tilde{X} \to X$ be a covering projection. Then, the pair $(\tilde{X}, \tilde{A})$ with $\tilde{A} = p^{-1}(A)$, is also an adjunction of n-cells.*

(ii) *Let (X, A) be an adjunction of n-cells, $n > 2$, and let $p : \tilde{A} \to A$ be a covering projection. Then, there are an adjunction of n-cells $(\tilde{X}, \tilde{A})$ and a* *covering projection $q:\tilde{X} \to X$, such that p is induced from q by the inclusion $A \to X$. In particular, if p is a universal covering projection, so is q.*

Proof (i) Consider $\Lambda = \pi(X \backslash A)$ as the index set for the n-cells of the adjunction (X, A). For every $\lambda \in \Lambda$, choose a characteristic map $\bar{c}_\lambda : B^n \to X$ for the cell e_λ. Then, take $\tilde{\Lambda} = \{(z, \lambda) \in \tilde{X} \times \Lambda : p(z) = \bar{c}_\lambda(e_0)\}$ and let $\bar{c}_{\tilde{\lambda}}$ denote the unique lifting of $\bar{c}_\lambda$ with $\bar{c}_{\tilde{\lambda}}(e_0) = z$, for any $\tilde{\lambda} = (z, \lambda) \in \tilde{\Lambda}$ (see Theorem A.8.5). Next, define $\bar{f}: B^n \times \tilde{\Lambda} \to \tilde{X}$ by $(s, \tilde{\lambda}) \mapsto \bar{c}_{\tilde{\lambda}}(s)$. The restriction $\bar{f}|S^{n-1} \times \tilde{\Lambda}$ factors through $\tilde{A}$ therefore, inducing a map $f: S^{n-1} \times \tilde{\Lambda} \to \tilde{A}$. It will be shown that $\tilde{X}$ may be viewed as being obtained from $\tilde{A}$ by adjoining $B^n \times \tilde{\Lambda}$ via f.

First, prove that every point $\tilde{x} \in \tilde{X} \backslash \tilde{A}$ corresponds to a unique point in $\mathring{B}^n \times \tilde{\Lambda}$. To this end, notice that $p(\tilde{x}) \notin A$, and so $p(\tilde{x}) = \bar{c}_\lambda(s)$, for a unique $\lambda \in \Lambda$ and a unique $s \in \mathring{B}^n$. Now, let W denote the line segment in B^n connecting s to e_0 and let $\omega : W \to \tilde{X}$ denote the unique lifting of $\bar{c}_\lambda | W$, with $\omega(s) = \tilde{x}$. Then, $\tilde{x} = \bar{c}_{\tilde{\lambda}}(s)$, with $\tilde{\lambda} = (\omega(e_0), \lambda)$.

Second, $\tilde{X}$ has the right topology. It will be shown that a subset $U \subset \tilde{X}$ is open if $U \cap \tilde{A}$ is open in $\tilde{A}$ and $\bar{c}_{\tilde{\lambda}}^{-1}(U)$ is open in B^n, for every $\tilde{\lambda} \in \tilde{\Lambda}$. Because p is a covering projection, there is an open cover $\{V_\gamma : \gamma \in \Gamma\}$ of $\tilde{X}$ such that the induced map $V_\gamma \to p(V_\gamma)$ is a homeomorphism and $p(V_\gamma)$ is open in X, for every $\gamma \in \Gamma$. Since U is open in $\tilde{X}$ iff $U \cap V_\gamma$ is open, for every γ, it suffices to assume $U \subset V_\gamma$, for some γ. But then, U is open iff $p(U)$ is open in X. Now $p(U) \cap A = p(U \cap \tilde{A})$ is open in A and $\bar{c}_\lambda^{-1}(p(U)) = \bigcup_z \bar{c}_{\tilde{\lambda}}^{-1}(U)$ where the union is taken over all z's such that $(z, \lambda) \in \tilde{\Lambda}$, is open in B^n, for every $\lambda \in \Lambda$; thus, because X has the final topology with respect to the inclusion of A and the characteristic maps $\bar{c}_\lambda$, the set $p(U)$ is open in X.

(ii) According to the condition on n, each attaching map for an n-cell of (X, A) has a simply connected domain, and so it has liftings to $\tilde{A}$. Use each of these liftings to attach an n-cell to $\tilde{A}$. The result is a space $\tilde{X}$ and the universal property of the attachings determines the covering projection q. □

Collaring

Whenever dealing with pairs (X, A) which are adjunctions of n-cells, $n > 0$, sometimes it is necessary to enlarge open sets of the subspace A to appropriate open sets of X. This can be done by the technique of 'collaring', which is described next.

Let $\bar{f} : \sqcup B_\lambda \to X$ be a characteristic map, and let $f : \sqcup S_\lambda \to A$ be the corresponding attaching map. Assume that every ball B_λ is just a copy of

B^n; thus, one can multiply any $s \in B_\lambda$ (viewed as a vector of $\mathbf{R}^n$) by a scalar $t \in I$; the product ts is still a point of B_λ. The $\bar{f}$*-collar* of a set $V \subset A$ is defined to be the subset

$$C_{\bar{f}}(V) = V \cup \bar{f}(\{ts : s \in f^{-1}(V), \quad \tfrac{1}{2} < t \leqslant 1\}).$$

The following is an immediate consequence of the definition.

Lemma 1.1.7 *Let (X, A) be an adjunction of n-cells, let $\bar{f}$ be a characteristic map for the adjunction, and let V be a subset of A. Then*

(i) $C_{\bar{f}}(V) \cap A = V$;

(ii) $\bar{f}^{-1}(C_{\bar{f}}(V)) = \{ts : s \in f^{-1}(V), \quad \tfrac{1}{2} < t \leqslant 1\}$;

(iii) *$C_{\bar{f}}(V)$ is open in X iff V is open in A;*

(iv) *if V is a closed subset of A, the closure of the $\bar{f}$-collar of V is the set*

$$\overline{C_{\bar{f}}(V)} = V \cup \bar{f}(\{ts : s \in f^{-1}(V), \quad \tfrac{1}{2} \leqslant t \leqslant 1\});$$

(v) *if e is an n-cell of (X, A), then $e \cap \overline{C_{\bar{f}}(V)} \neq \varnothing$ iff $e \cap C_{\bar{f}}(V) \neq \varnothing$ iff $\bar{e} \cap V \neq \varnothing$;*

(vi) *$C_{\bar{f}}(V)$ contains V as a strong deformation retract;*

Moreover, if (V_γ) is a locally finite family of subsets of A (respectively, a family of pairwise disjoint subsets of A), then

(vii) *$(C_{\bar{f}}(V_\gamma))$ is a locally finite family of subsets of X (respectively, a family of pairwise disjoint subsets of X).* □

The next result requires a little work.

Lemma 1.1.8 *Let (X, A) be an adjunction of n-cells and $\bar{f}$ be a characteristic map for the adjunction. If $V \subset A$ is open or closed in A, then*

$$C_{\bar{f}}(V) = V \sqcup_g (f^{-1}(V) \times (\tfrac{1}{2}, 1])$$

where f is the attaching map corresponding to $\bar{f}$ and $g : f^{-1}(V) \to V$ is the map induced by f.

Proof Assume first that V is open in A. Then, because of Lemma 1.1.7 (iii), $C_{\bar{f}}(V)$ is open in X; the stated result now follows by application of the restriction law (L4) of adjunction spaces, and parts (i) and (ii) of the previous lemma. In particular, notice that

$$C_{\bar{f}}(A) = A \sqcup_f (f^{-1}(A) \times (\tfrac{1}{2}, 1]).$$

Now if V is closed in A, then $C_{\bar{f}}(V)$ is closed in $C_{\bar{f}}(A)$, and so the statement follows again by (L4). □

Corollary 1.1.9 *If V is an open or closed subspace of A, then the inclusion $V \rightarrowtail C_{\bar{f}}(V)$ is a closed cofibration.* □

The fact that the characteristic maps are not unique might be quite advantageous; indeed, it permits the choice of the 'right coordinates' for a variety of purposes, as proved by the next proposition.

Proposition 1.1.10 *Let (X, A) be an adjunction of n-cells, V be a closed set of A and U be an open subset of X containing V. Then there is a characteristic map $\bar{f}$ for the adjunction such that the closure $\overline{C_{\bar{f}}(V)}$ is still contained in U.*

Proof Choose arbitrarily a characteristic map $\tilde{f} : \sqcup B_e \to X$, where the index e runs through all the n-cells of the adjunction. The map $\tilde{f}$ determines an attaching map $f : S_e \to A$ whose restriction to a sphere S_e will be denoted by f_e. The objective is to construct cellwise a 'transformation of coordinates', which keeps the attaching map f invariant. Notice that $\tilde{f}$ must be modified only for cells e, such that

$$(*) \quad \tilde{f}(\{ts : s \in S_e, f(s) \in V, \tfrac{1}{2} \leqslant t \leqslant 1\}) \not\subset U.$$

Let e be such a cell. Then $V_e = f_e^{-1}(V)$ is non-empty and $\tilde{f}(B_e)$ is not completely contained in U. Hence the set $U_e = B_e \backslash \tilde{f}^{-1}(U)$ is a non-empty closed subset of B_e which does not meet the closed set V_e. The distance δ_e between the closed sets U_e and V_e is defined and different from 0, because these two closed sets are both compact subsets of a metric space. It is easy to conclude from $(*)$ that $\delta_e \leqslant \frac{1}{2}$. Next, select a homeomorphism $h_e : B_e \to B_e$ which coincides with the identity map on the boundary of B_e and shrinks the ball $\{s \in B_e : |s| \leqslant 1 - \delta_e\}$ radially into the ball $\{s \in B_e : |s| \leqslant \frac{1}{3}\}$; then define

$$\bar{f} | B_e = \tilde{f} | B_e \circ h_e^{-1}.$$

This completes the construction of the desired characteristic map $\bar{f}$. □

Exercises

1. Let (Y, D) be an adjunction of n-cells and let A be a contractible space. Show that any map $f : D \to A$ can be extended over Y.
2. Let $M(\pi, n)$ be a Moore space of type (π, n). Show that $M(\pi, n)$ is up to homotopy independent of the choice of the basis Γ selected for the kernel of the canonical homomorphism $FA(\pi) \to \pi$; show also that $M(\pi, n)$ does not depend on the choice of the representatives $f_\gamma : S^n \to \vee_\pi S^n$ (see Example 10).