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Introduction

In this book we are concerned with the ways in which surfaces or interfaces modify the properties of solids and liquids. Various different effects may be identified. First, there may be a modification to the equilibrium configuration in a medium close to a surface; this is known as *surface reconstruction*. For example, the atoms near to a surface may have a different crystallographic arrangement compared with those in the bulk, or they may be disordered. Another example is a ferromagnetic solid, in which the interactions between the magnetic moments at the surface may differ from those in the bulk, leading to a different value of the magnetisation. Clearly this type of effect may be temperature dependent, and it is particularly relevant when there is a phase transition (e.g. close to the Curie temperature in a ferromagnet). Second, the *excitations* within the system (such as the phonons in the lattice dynamics of a crystal or the magnons of a ferromagnet) are modified by a surface. In an infinite medium the bulk (or volume) excitations are characterised by an amplitude that varies in a wave-like fashion in three dimensions. When surfaces are present the bulk excitations are required to satisfy appropriate boundary conditions. Consequently there will, in general, be changes to the density of states of the bulk excitations and, in some cases to the bulk excitation frequencies. However, a more interesting effect of a surface on the excitation spectrum is that it can give rise to localised surface excitations (e.g. surface phonons or surface magnons). By contrast with bulk excitations, the surface excitations are wave-like for propagation parallel to the surface (or interface) and have an amplitude that decays with distance away from the surface.

    As indicated by the title of this book, we are primarily concerned with the *dynamic* properties of finite media, i.e. with the excitations. For the most part, we include discussion of the static properties of surfaces

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(the surface reconstruction) only to the extent that these may influence the phase behaviour and the nature of the excitation spectrum. The general characteristics of bulk and surface excitations, together with their symmetry aspects, are described in §1.1, and §1.2 deals with surface reconstruction. This is followed in §1.3 by a brief account of some of the theoretical methods applied to bulk and surface excitations, while §1.4 contains a preliminary survey of the experimental methods. Specific types of excitations and/or specific surface structures are dealt with in the later chapters.

1.1 Excitations in crystals

We now introduce some of the general properties of surface excitations, stressing the symmetry aspects. It is instructive to do this by comparing and contrasting them with bulk excitations in infinite media, for which the concepts are more familiar. The results for bulk excitations will be summarised first; details are to be found in any of the standard textbooks on solid-state physics (e.g. Ziman 1972; Elliott and Gibson 1974; Ashcroft and Mermin 1976; Kittel 1986).

1.1.1 Bulk excitations

An ideal crystal can be described as a *basis* of one or more atoms or ions located at each point of a *lattice*. A lattice is an infinitely-extended regular periodic array of points in space. All lattice points are equivalent, and the system possesses *translational symmetry*. The lattice may be defined in terms of three fundamental translation vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, which are non-coplanar. If a translation is made through any vector that is a combination of integral multiples of these basic vectors the crystal appears unchanged. The end points of such vectors $\mathbf{R}$, defined by

$$\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 \tag{1.1}$$

with n_1, n_2 and n_3 integers, form the *space lattice*. In addition to the translation operations, there may also in general be other symmetry operations, such as certain rotations and reflections, which leave the crystal apparently unchanged. All such symmetry operations must leave both the space lattice and basis unchanged.

An important part is played by the *unit cell*, defined as the smallest volume based on one lattice point such that the whole of space is filled by repetitions of the unit cell at each lattice point. The specification of the unit cell is not unique: one possible choice would be the parallelepiped subtended by the basic vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, with the cell arbitrarily centred on one of the atomic positions. Some choices of unit cell are

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illustrated in Fig. 1.1, which depicts for ease of representation a two-dimensional lattice with basic vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Two examples of parallelograms as the unit cell are shown, both with sides defined by $\mathbf{a}_1$ and $\mathbf{a}_2$ but centred at different positions on the space lattice. A hexagonal unit cell also is indicated, obtained by drawing the perpendicular bisectors of the lattice vectors from a central point to the nearby equivalent sites (a construction known generally as the *Wigner–Seitz cell*).

The translational symmetry of the crystal structure implies that various position-dependent physical quantities, such as the electron density or the electrostatic potential, are the same within each unit cell. These quantities must be multiply-periodic functions satisfying

$$f(\mathbf{r} + \mathbf{R}) = f(\mathbf{r}) \tag{1.2}$$

for all points $\mathbf{r}$ in space and for all vectors $\mathbf{R}$ defined by (1.1). In one dimension it is a well-known mathematical result that a periodic function can be expanded as a Fourier series of complex exponentials. The analogous result in three dimensions enables us to write

$$f(\mathbf{r}) = \sum_{\mathbf{Q}} F(\mathbf{Q}) \exp(i\mathbf{Q} \cdot \mathbf{r}) \tag{1.3}$$

where the vectors $\mathbf{Q}$ must satisfy

$$\exp(i\mathbf{Q} \cdot \mathbf{R}) = 1 \tag{1.4}$$

for all lattice vectors $\mathbf{R}$. The end points of all vectors $\mathbf{Q}$ satisfying (1.4) form a lattice known as the *reciprocal lattice* (since $\mathbf{Q}$ has the dimension of a reciprocal length). The reciprocal lattice can be generated from the basic vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij} \qquad (i, j = 1, 2, 3) \tag{1.5}$$

Fig. 1.1 General oblique lattice in two dimensions showing the basic vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Three possible forms of the unit cell are indicated – two are parallelograms with different centres, and the other (the Wigner–Seitz cell) a hexagon obtained by drawing the perpendicular bisectors of the lines from a central point to nearby lattice points.
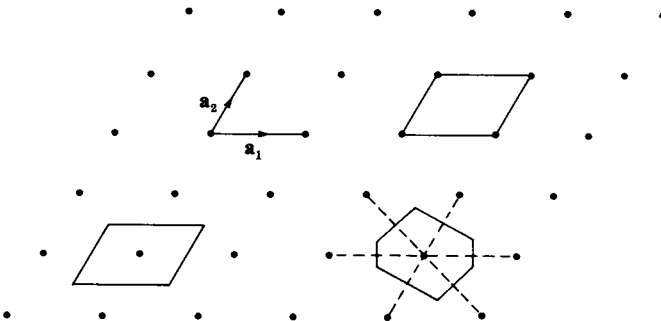
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where δ_{ij} is the Kronecker delta (defined by $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$). Explicitly, the definitions are

$$\mathbf{b}_1 = \frac{2\pi \mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} \qquad \mathbf{b}_2 = \frac{2\pi \mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} \qquad \mathbf{b}_3 = \frac{2\pi \mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} \qquad (1.6)$$

A general reciprocal lattice vector $\mathbf{Q}$ then takes the form

$$\mathbf{Q} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + v_3 \mathbf{b}_3 \qquad (1.7)$$

where v_1, v_2 and v_3 are integers. The proof that (1.6) and (1.7) lead to the property (1.4) is left to the reader (see Problem 1.1 at the end of this chapter).

The unit cell of the reciprocal lattice is conventionally obtained using the Wigner–Seitz construction described earlier; it is known as the *Brillouin zone*. Its volume is the same as the parallelepiped formed by the basic vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, which can be shown from (1.6) to be $8\pi^3/V_0$ where V_0 is the volume of the unit cell in real space. The easiest example is a simple cubic space lattice, for which the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ may be taken as

$$\mathbf{a}_1 = a(1, 0, 0) \qquad \mathbf{a}_2 = a(0, 1, 0) \qquad \mathbf{a}_3 = a(0, 0, 1) \qquad (1.8)$$

where a is the lattice parameter (the nearest-neighbour separation). The basic vectors of the reciprocal lattice are easily found to be

$$\mathbf{b}_1 = \frac{2\pi}{a}(1, 0, 0) \qquad \mathbf{b}_2 = \frac{2\pi}{a}(0, 1, 0) \qquad \mathbf{b}_3 = \frac{2\pi}{a}(0, 0, 1) \qquad (1.9)$$

Hence the reciprocal lattice is also simple cubic in this case. It is fairly straightforward to show that the reciprocal lattice of a body-centred cubic (bcc) space lattice is face-centred cubic (fcc) and the reciprocal lattice of a fcc space lattice is bcc. The details are given in solid-state textbooks.

We are now in a position to discuss how the elementary excitations of a crystal are influenced by the symmetry. The important property is embodied in a result that is generally known to solid-state physicists as *Bloch's theorem* (Bloch 1928); it is also related to Floquet's theorem in mathematics (e.g. see Whittaker and Watson 1963). Here we outline a simple proof applicable to any type of excitation in a crystal. Examples of excitations that we shall consider in later chapters include lattice vibrations (phonons), spin waves (magnons), electronic modes (such as plasmons), and so forth. All of them are excitations of the whole region, rather than localised excitations of particular atoms, and this leads to common symmetry features.

A consequence of the periodicity of the crystal lattice is that a position-dependent quantity, such as the Hamiltonian operator $H(\mathbf{r})$, is unaltered by a translation through the vector $\mathbf{R}$ in (1.1). For example,

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the Hamiltonian might take the form $-(\hbar^2/2m)\nabla^2 + V(\mathbf{r})$ appropriate to an electron (mass m) in a periodic potential $V(\mathbf{r})$. The translational invariance property of the Hamiltonian can be stated as $H(\mathbf{r} + \mathbf{R}) = H(\mathbf{r})$. It follows that the corresponding Schrödinger's equation

$$H(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{1.10}$$

must be invariant under the translation $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{R}$. Hence, if $\psi(\mathbf{r})$ denotes the wavefunction of a stationary state (with energy eigenvalue E), then $\psi(\mathbf{r} + \mathbf{R})$ is also a solution describing the same state of the system. This implies that the two functions must be related by a multiplicative factor, and we write

$$\psi(\mathbf{r} + \mathbf{R}) = c\psi(\mathbf{r}) \tag{1.11}$$

It is evident that c must have unit modulus, otherwise the wavefunction would tend to infinity if the translation through $\mathbf{R}$ (or $-\mathbf{R}$) were repeated indefinitely. Hence c must be expressible as

$$c = \exp(i\mathbf{q} \cdot \mathbf{R}) \tag{1.12}$$

where $\mathbf{q}$ is an *arbitrary* (real) constant vector: it has dimensions of reciprocal length. The general form of the wavefunction having the above property is

$$\psi(\mathbf{r}) = \exp(i\mathbf{q} \cdot \mathbf{r})U_{\mathbf{q}}(\mathbf{r}) \tag{1.13}$$

where $U_{\mathbf{q}}(\mathbf{r})$ is a periodic function: $U_{\mathbf{q}}(\mathbf{r} + \mathbf{R}) = U_{\mathbf{q}}(\mathbf{r})$. The overall phase factor $\exp(i\mathbf{q} \cdot \mathbf{r})$ gives a plane-wave variation to the wavefunction in accordance with the properties (1.11) and (1.12) deduced from the translational symmetry of the lattice.

An alternative (and more rigorous) proof of Bloch's theorem (1.13) can be formulated using group theory (e.g. see Wherrett 1986). For convenience we have expressed the result in terms of the wavefunction $\psi(\mathbf{r})$, as would be the case for electronic states. More generally, Bloch's theorem would be written in terms of whatever variable is used to describe the amplitude of the excitation (e.g. an atomic displacement in the case of phonons).

The wavevector $\mathbf{q}$ is clearly not unique: values differing by any reciprocal lattice vector $\mathbf{Q}$ as defined in (1.7) would equally satisfy (1.12) because $\exp[i(\mathbf{q} + \mathbf{Q}) \cdot \mathbf{R}] = \exp(i\mathbf{q} \cdot \mathbf{R})$. In general, the energy eigenvalue in (1.10) will depend on $\mathbf{q}$, and it is periodic in the reciprocal lattice. Denoting $E = \hbar\omega(\mathbf{q})$, where $\omega(\mathbf{q})$ is the excitation frequency, we must therefore have

$$\omega(\mathbf{q} + \mathbf{Q}) = \omega(\mathbf{q}) \tag{1.14}$$

For an infinite crystal $\mathbf{q}$ can take any real value but, because of the above property, it is unnecessary to define it outside the first Brillouin zone, defined to be the Brillouin zone centred at $\mathbf{q} = 0$. For example, earlier in

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this section we showed that the Brillouin zone of a simple cubic lattice is a cube with sides of dimension $2\pi/a$. Hence it is sufficient in this case to take $\mathbf{q}$ values corresponding to

$$-\pi/a < q_x \leqslant \pi/a$$
$$-\pi/a < q_y \leqslant \pi/a \qquad (1.15)$$
$$-\pi/a < q_z \leqslant \pi/a$$

In a finite but macroscopically large crystal, the number of excitations (and hence the number of allowed values of $\mathbf{q}$ in the Brillouin zone) is finite. For ease of discussion, suppose the crystal has a simple cubic structure and that the sample is macroscopically large forming a cube of side Na. Strictly in such problems it would become necessary to introduce boundary conditions describing the surfaces (as discussed later). However, if N is very large and only bulk properties of the crystal are being examined, this difficulty is often circumvented by the use of cyclic (or Born–von Karman) boundary conditions. To illustrate this we take a one-dimensional (1D) case of a 'crystal' of length Na, with the ends joined to form a ring. The condition that

$$\psi(z + Na) = \psi(z) \qquad (1.16)$$

where z is a position coordinate, follows from the requirement that the wavefunction is single-valued. However, the 1D case of (1.11) is

$$\psi(z + a) = c\psi(z) \qquad (1.17)$$

for a symmetry translation through a. Thus it follows that $c^N = 1$, which has the roots $c = \exp(iq_z a)$ where

$$q_z = \frac{2\pi m_z}{Na} \qquad (1.18)$$

and m_z is an integer. Within the first Brillouin zone, m_z will take the N consecutive integer values in the range $-\frac{1}{2}N < m_z \leqslant \frac{1}{2}N$ (if N is even). The generalisation of the above argument to 3D is obvious: each component of the vector $\mathbf{q}$ will be given by an expression analogous to (1.18). Thus we obtain for the allowed wavevectors a set of uniformly distributed points in $\mathbf{q}$-space. The 'volume' per allowed wavevector is a constant, namely $(2\pi/Na)^3$. Since N is supposed to be very large this distribution is effectively continuous, and it is permissible to convert summations over discrete $\mathbf{q}$ values to integrations over continuous variables by the prescription

$$\sum_{\mathbf{q}} \rightarrow \frac{V}{(2\pi)^3} \iiint \mathrm{d}q_x\, \mathrm{d}q_y\, \mathrm{d}q_z \qquad (1.19)$$

where $V = (Na)^3$ is the volume of the crystal.

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1.1.2 Surface excitations

In the presence of one or more surfaces, many of the symmetry arguments presented in the preceding subsection need to be modified. To illustrate this we restrict attention initially to the effect of a single plane surface. Specifically we consider the situation represented in Fig. 1.2, where the surface is taken to lie in the plane $z = 0$ of a Cartesian coordinate system. The thickness of the crystal and its dimensions in the x and y directions will be taken as sufficiently large (effectively infinite) that the crystal may be regarded as filling the half-space $z \leqslant 0$. For the present we ignore any rearrangement or distortion of the surface atoms (i.e. surface reconstruction) and deal with an 'idealised' surface.

If the orientation of the surface relative to the crystal axes is such that two of the basic translation vectors ($\mathbf{a}_1$ and $\mathbf{a}_2$, say) are parallel to the surface, while $\mathbf{a}_3$ is not, then the set of translations $\mathbf{R}_\parallel$ defined by the vectors

$$\mathbf{R}_\parallel = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \qquad (1.20)$$

with n_1 and n_2 integers, are symmetry operations of the crystal. However, translation operations involving $\mathbf{a}_3$, such as those given by (1.1) with $n_3 \neq 0$, are *not* symmetry operations: they do not leave the crystal apparently unchanged, because they connect points that have different positions relative to the surface plane. The end points of the set of vectors $\mathbf{R}_\parallel$ form the 2D space lattice. There are in fact only five space lattices (called *Bravais lattices*) in 2D, compared with fourteen in 3D. These are depicted in Fig. 1.3.
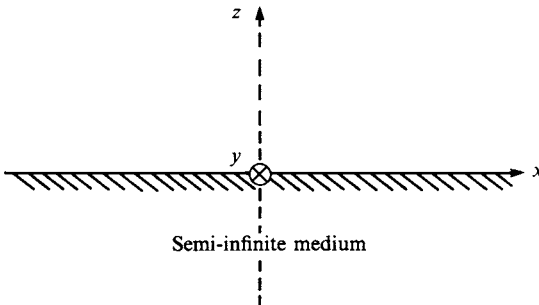
For the corresponding 2D reciprocal lattice there are just two basic vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. They are defined as the vectors in the xy plane that satisfy

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij} \qquad (i, j = 1, 2) \qquad (1.21)$$

A general 2D reciprocal lattice vector $\mathbf{Q}_\parallel$ can then be expressed as

$$\mathbf{Q}_\parallel = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 \qquad (1.22)$$

Fig. 1.2 Choice of coordinate axes for a semi-infinite medium.

Semi-infinite medium

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with v_1 and v_2 integers. The Wigner–Seitz unit cell of the 2D reciprocal lattice defines the 2D Brillouin zone: its area is equal to $4\pi^2/A_0$ where A_0 is the area of the unit cell for the lattice planes parallel to the surface.

The periodicity property (1.2) satisfied by any position-dependent physical quantity $f(\mathbf{r})$ in an infinite 3D system no longer holds in the present case because there is translational symmetry only in the xy plane. The more limited property applicable to the semi-infinite crystal is that

$$f(\mathbf{r}_\parallel + \mathbf{R}_\parallel, z) = f(\mathbf{r}_\parallel, z) \tag{1.23}$$

where we denote $\mathbf{r} = (\mathbf{r}_\parallel, z)$ for the 3D position vector and $\mathbf{r}_\parallel = (x, y)$. From the properties of the 2D reciprocal lattice vectors and from the lattice periodicity in the xy plane we may define a 2D Fourier series expansion by
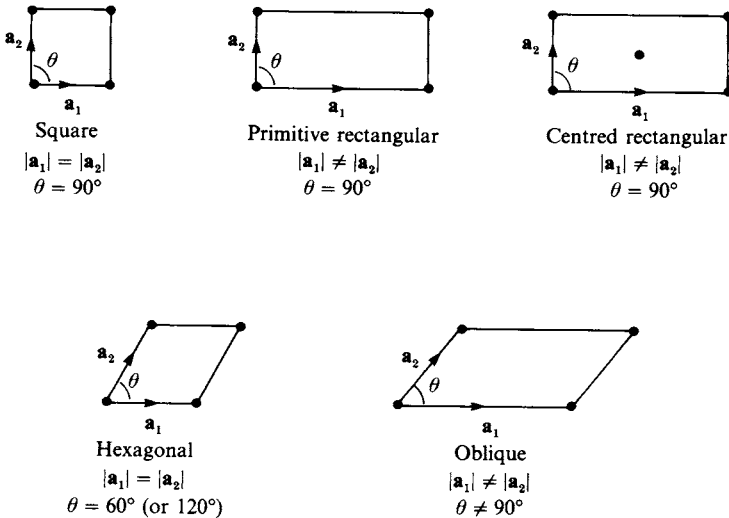
$$f(\mathbf{r}_\parallel, z) = \sum_{\mathbf{Q}_\parallel} F(\mathbf{Q}_\parallel, z) \exp(i\mathbf{Q}_\parallel \cdot \mathbf{r}_\parallel) \tag{1.24}$$

This is the analogue of (1.3) for the infinite 3D case.

The proof of Bloch's theorem also requires modification in the presence of a surface. We may follow the same type of argument as in the preceding section: equation (1.11) must be replaced by $\psi(\mathbf{r}_\parallel + \mathbf{R}_\parallel, z) = c\psi(\mathbf{r}_\parallel, z)$ because $\mathbf{R}_\parallel$ is now the general operation of translation symmetry. This leads to the conclusion that

$$c = \exp(i\mathbf{q}_\parallel \cdot \mathbf{R}_\parallel)$$

Fig. 1.3 The five Bravais lattices in two dimensions, defined in terms of the basic vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Square
$|\mathbf{a}_1| = |\mathbf{a}_2|$
$\theta = 90°$

Primitive rectangular
$|\mathbf{a}_1| \neq |\mathbf{a}_2|$
$\theta = 90°$

Centred rectangular
$|\mathbf{a}_1| \neq |\mathbf{a}_2|$
$\theta = 90°$

Hexagonal
$|\mathbf{a}_1| = |\mathbf{a}_2|$
$\theta = 60°$ (or $120°$)

Oblique
$|\mathbf{a}_1| \neq |\mathbf{a}_2|$
$\theta \neq 90°$

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where $\mathbf{q}_\| = (q_x, q_y)$ is a *real* 2D vector, and the 2D analogue of Bloch's theorem becomes

$$\psi(\mathbf{r}_\|, z) = \exp(i\mathbf{q}_\| \cdot \mathbf{r}_\|) U_{\mathbf{q}_\|}(\mathbf{r}_\|, z) \tag{1.25}$$

Here $U_{\mathbf{q}_\|}(\mathbf{r}_\|, z)$ has the 2D periodicity property that $U_{\mathbf{q}_\|}(\mathbf{r}_\| + \mathbf{R}_\|, z) = U_{\mathbf{q}_\|}(\mathbf{r}_\|, z)$.

It is not possible to use symmetry arguments to make a rigorous statement about the dependence of $U_{\mathbf{q}_\|}(\mathbf{r}_\|, z)$ on the coordinate z perpendicular to the surface. The procedure for deducing the z dependence in any specific problem is frequently to obtain a differential equation (or, in some cases, a finite-difference equation) for $U_{\mathbf{q}_\|}(\mathbf{r}_\|, z)$; this would be Schrödinger's equation in the case of electronic states or the appropriate equation of motion in the case of other excitations. The differential equation must be solved subject to the relevant boundary conditions; in the present example, these are at the surface $z = 0$ and at $z \to -\infty$. It is usually found that this differential equation has more admissible solutions than is the case in the corresponding infinite crystal problem. Suppose that the differential equation for the semi-infinite medium admits solutions of the form $\exp(iq_z^{(j)}z)$ where $q_z^{(j)}$ may be *complex* and take a series of values (labelled by j) determined from the differential equation. The solution for $U_{\mathbf{q}_\|}(\mathbf{r}_\|, z)$ would be formed from a linear combination of such terms:

$$U_{\mathbf{q}_\|}(\mathbf{r}_\|, z) = \sum_j B_{\mathbf{q}_\|}^{(j)}(\mathbf{r}_\|) \exp(iq_z^{(j)}z) \tag{1.26}$$

where $B_{\mathbf{q}_\|}^{(j)}(\mathbf{r}_\|)$ is an amplitude factor. Because the right-hand side of (1.26) must remain finite at large distances from the surface ($z \to -\infty$), we *either* have $q_z^{(j)}$ is real *or* $q_z^{(j)}$ is complex with $\mathrm{Im}(q_z^{(j)}) < 0$.

The former possibility corresponds to a *bulk excitation*, since $\exp(iq_z^{(j)}z)$ has a constant modulus (equal to unity) for all z. The quantity $q_z^{(j)}$ is just the third component of the 3D wavevector $\mathbf{q} = (\mathbf{q}_\|, q_z^{(j)})$ describing the propagation of the excitation. The bulk excitation is affected by the surface in that it must satisfy a boundary condition at $z = 0$, and this will enter into the calculation of the corresponding B-coefficient in (1.26).

The second of the above possibilities, i.e. $\mathrm{Im}(q_z^{(j)}) < 0$, corresponds to a *surface excitation* localised near the $z = 0$ surface, because $\exp(iq_z^{(j)}z) \to 0$ as $z \to -\infty$ within the crystal. If $q_z^{(j)}$ is pure imaginary we may denote $q_z^{(j)} = -i\kappa$ (with κ real and positive), and the surface excitation has a simple exponential decay proportion to $\exp(\kappa z)$ as $z \to -\infty$. The attenuation length (or decay length) is $1/\kappa$. More generally, $q_z^{(j)}$ is complex and this corresponds to an excitation that oscillates within an exponentially decaying envelope function as $z \to -\infty$. Hence, in such cases, the surface excitation can be characterised by a 2D wavevector $\mathbf{q}_\|$

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describing its propagation parallel to the surface and by a decaying amplitude in the direction perpendicular to the surface.

The concepts of bulk and surface excitations may be illustrated by considering some aspects of bulk and surface waves in a semi-infinite isotropic elastic medium. This problem was first examined by Rayleigh (1887) and we return to it in detail in Chapter 2. The general equation of motion is the wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \nabla^2 u = 0 \tag{1.27}$$

where u denotes any component of the vectors $\mathbf{u}_L$ and $\mathbf{u}_T$ for longitudinal and transverse displacements in the elastic medium, and $v = v_L$ or v_T is the corresponding velocity. Assuming coordinate axes as in Fig. 1.2 we seek a plane-wave solution to (1.27) propagating parallel to the surface with wavevector $\mathbf{q}_\parallel$ and frequency ω:

$$u(\mathbf{r}, t) = \exp(i\mathbf{q}_\parallel \cdot \mathbf{r}_\parallel) f(z) \exp(-i\omega t) \tag{1.28}$$

On substituting (1.28) into (1.27) the equation for $f(z)$ is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - \left(q_\parallel^2 - \frac{\omega^2}{v^2} \right) f = 0 \tag{1.29}$$

It can now be seen that there are two types of solution for $f(z)$. If $q_\parallel^2 < \omega^2/v^2$ we have

$$f(z) = B_1 \exp(iq_z z) + B_2 \exp(-iq_z z) \tag{1.30}$$

where

$$q_z = \left(\frac{\omega^2}{v^2} - q^2 \right)^{1/2} \tag{1.31}$$

This describes a bulk wave. The two terms in (1.30) describe a wave propagating towards the surface and a reflected wave. The other type of solution of (1.29) occurs when $q_\parallel^2 > \omega^2/v^2$ and it corresponds to

$$f(z) = B_3 \exp(\kappa z) \tag{1.32}$$

where

$$\kappa = \left(q^2 - \frac{\omega^2}{v^2} \right)^{1/2} \tag{1.33}$$

This describes a surface wave decaying with distance from the surface (recall that $z < 0$ within the solid). To proceed further with this calculation one would need to form the total displacement $(\mathbf{u}_L + \mathbf{u}_T)$ and then bring in the boundary conditions at $z = 0$. The details are given in §2.3.1.

In the discussion so far, we have restricted attention to the simplest case of a semi-infinite medium, so that only one surface is involved. The extension to a parallel-sided slab (or film) of finite thickness involves

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