GEOMETRY AND INTERPOLATION OF CURVES AND SURFACES

This text takes a practical, step-by-step approach to algebraic curves and surface interpolation motivated by the understanding of the many practical applications in engineering analysis, approximation, and curve-plotting problems. Because of its usefulness for computing, the algebraic approach is the main theme, but a brief discussion of the synthetic approach is also presented as a way of gaining additional insight before proceeding with the algebraic manipulation.

The authors start with simple interpolation, including splines. In an intuitive fashion they extend these simple procedures to the production of conic sections. They then introduce projective coordinates as tools for dealing with higher-order curves and such important concepts as singular points. They present many applications and concrete examples, including an analysis of the rational and polynomial cubics, parabolic interpolation, geometric approximation, and the numerical solution of trajectory problems. In the final chapter they apply the basic theory to the construction of finite-element basis functions and surface interpolants over nonregular shapes and discuss the simple cases of the Steiner surface and the cubic surface.

Professionals, students, and researchers in applied mathematics, solid modeling, graphics, robotics, and engineering design and analysis will find this a useful reference.

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Preface

Background

The text we present here is the outcome of many years of research and study, by the authors, on a variety of applied problems. Our primary training was in numerical analysis, and from that viewpoint we pursued some problems in finite-element methods, an important area of engineering analysis. Soon we began to suspect that some aspects of geometry might be useful in our research, but we did not know which ones. Geometry seemed rather confusing, and there were many different kinds and much material. We were overwhelmed. Each branch of the subject seemed an enormous study, and we wanted to cut down our work to the areas that, we hoped, would be most relevant to our applied interests. We hoped for a simple program that would yield quick results in only a few days of study. It was not that simple.

However, as we studied we discovered some remarkable things. We found that the subject was much more beautiful than we had imagined, and we were drawn towards it. We also found that, as we explored, we stumbled across answers to applied problems that had defeated mathematicians for years. We even found that the geometry made significant contributions to areas we had not initially set out to explore, and we were drawn into research on curve interpolation, geometrical approximation, computer graphics, and finite-difference methods.

This book is the story of our journey, though we have organized it in a well-structured way and not as the meandering network of pathways that we actually trod.

The book could serve as a complete text for a year-long course on algebraic geometry and applications, but shorter courses, using only selected parts of the text, are equally possible. We will make some specific suggestions later in the Preface, but first, let us set the scene by giving an example.

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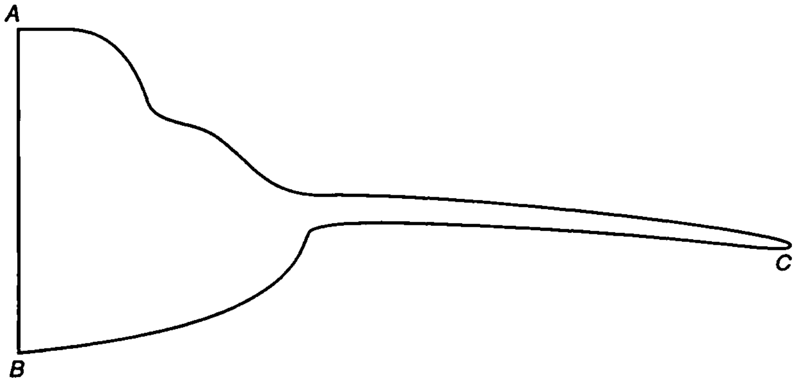

Figure 0.1. Aircraft cross section.

A Real-Life Example

Let us imagine that we have been posed the problem of giving a mathematical description of a curve like the one represented in Figure 0.1. There are two common situations in which such a problem may arise. The first situation is what we shall refer to as the *design problem*. In this situation we are not given any curve, but have to create one. The criteria determining whether or not the curve we created is a "good" one are concerned with constraints on the entire problem. For example, in aircraft design these criteria may be connected with loading factors, lift, strength, and so on. In other environments, such as graphic art designs for the motion-picture industry or as used by clothing manufacturers, the criteria may be nothing more than the creation of esthetically pleasing curves. The second situation, which we shall refer to as the *approximation problem*, is the case where we are provided with a specific curve – perhaps a draughtsman's sketch, perhaps a set of points – and we have to produce a curve that is "close" to the given curve. Many problems involve aspects of both design and approximation. In any of the above situations it will be necessary to produce some curve and to know what curve has been produced. It may not be necessary to have a closed-form mathematical expression for the equation of the curve, provided that there is an algorithm for reproducing the points on the curve to within some specified tolerance. In most cases, however, some closed-form expression is indeed sought, even though in many of them, as when machining a part using a numerically controlled machine, the curves are reapproximated and finally represented as a set of discrete points.

Suppose we want to approximate a curve similar to the one shown in Figure 0.1. In addition, we may have been given the distance AB, the distance from C to the line AB, and perhaps a few other important measurements, but otherwise we know nothing about the curve apart from its qualitative behavior as displayed in the drawing, that is, we have visual rather than mathematical information. We must mathematically produce a curve which is, in some sense, close to the given curve and displays the same qualitative behavior. How should we set about solving such a problem? Our first thought might be to choose some function $y = f(x)$, then to select a set of x-values $\{x_i : i = 1, 2, \ldots, n\}$ and force the corresponding y-values $f(x_i)$ to be such that the points $(x_i, f(x_i))$, for $i = 1, 2, \ldots, n$, lie on the curve. But what are x and y? Our drawing was not supplied with coordinate axes, and any imposition of axes will be somewhat arbitrary. This is clearly undesirable, since our approximation would depend not only on the set of points $\{(x_i, f(x_i))\}$, but also on the particular orientation of the axes. Moreover, since functions are single-valued, there is no function that can approximate both the upper part and the lower part of the curve in Figure 0.1. We could use the common, if ponderous, technique of thinking of the curve as being composed of several pieces, each selected in such a way that it does, in fact, represent a function in its own local coordinate system. However, if we divide the curve into pieces, then it may be that a piecewise approximation technique, such as using a conic to approximate each piece, would yield satisfactory results. Such as technique is often used, and we will describe it in Chapter 2. Alternatively, we could abandon the "$y = f(x)$" form of approximation and use a curve representation that permits curves that can indeed "go round corners."

Two such representations come to mind, namely, a curve defined parametrically, that is, in the form $x = X(t)$, $y = Y(t)$, and a curve given implicitly, that is, in the form $f(x, y) = 0$. The simplest parametric form is polynomials. We may think that we are now in an ideal situation, for we now have a simple representation of the approximating curve, although, as we shall discuss in Chapter 5, such curves in general possess loops and cusps and may also display unwanted inflection points. This behavior would clearly be undesirable, and any method of approximation that uses such curves ought to pay due attention to such potentially troublesome qualitative behavior. Furthermore, there is the difficulty, when using parametric curves, of quantifying what is an appropriate meaning for "close." The usual norms are pertinent only to function approximation and are therefore not immediately applicable to the situation of curve approximation using parametric curves. Some of the special techniques that can be used to measure the distance between curves will be discussed in Chapter 6.

Were we to choose an implicit definition for the approximating curve, then the questions of closeness of approximation that arise in parametric approximation still prevail. As to the form of the function $f(x, y)$, it would be reasonable, as a first attempt, to choose polynomials, the simplest nontrivial case being that of a quadratic polynomial, hence giving us a conic approximation. Even a cursory glance at the curve in Figure 0.1 would indicate that no single conic could be expected to give a reasonable approximation to this curve. A disadvantage of using implicitly defined curves is the difficulty in plotting the final curve. With curves that are given as functions or with parametrically defined curves no such problems arise, but the plotting of implicitly defined curves involves, in general, the solution of a nonlinear equation. We shall return to this question in Chapter 6.

Further questions come to mind. Is there a place for approximation with implicitly defined curves other than conics? Is the choice of parameter important in parametric approximation? Is there a connection between parametric approximation and approximation using implicitly defined curves? How can one control curvature? Some of these questions and the problems cited above will be discussed later. In the meantime, we should notice that in each of the above postulated situations our proposed technique would involve some form of interpolation. It is very important, therefore, that we develop a clear understanding of simple interpolation before proceeding to other and more sophisticated techniques, so many of which rely heavily upon it.

The key step in the finite-element method is the use of a local basis whose construction is the solution of an interpolation problem, and most of the curve and surface design in CAD/CAM (computer-aided design/computer-aided manufacture) is, in fact, interpolation, although here, as in some other applications, there are some "nonstandard" interpolation problems. Interpolation of data is the topic under discussion in Chapter 1. Many of the curve interpolation problems will be discussed in Chapter 6, and surface interpolation will be discussed in Chapter 7.

Course Suggestions

Elementary Geometry

Many students have met conics in high school or during a first-year college math course. For such students a course starting with Chapter 2 might be suitable. If the instructor desires some links with interpolation, Sections 1.6.1 through 1.7.2 from Chapter 1 will be instructive.

At this point the instructor must make another choice. If some elementary synthetic geometry is desired, then Chapter 3 should be studied before proceeding with Chapter 4. However, Chapter 3 can be omitted and the student can proceed directly to Chapter 4. The latter chapter is a basic introduction to algebraic projective geometry and should not be omitted. The approach is, essentially, a transformational approach and is consistent with the current way of approaching coordinate geometry in American high schools. The material, however, goes well beyond that covered in high schools.

This outline will suffice for a simple course on algebraic geometry at an elementary level. If time permits, some examples from Chapter 6 (perhaps from Section 6.7 or 6.8) might provide an interesting applications section to the course.

Algebraic Curves

For the more advanced student or the student specifically interested in algebraic curves and with some pertinent previous background, Chapter 2 can be covered quickly, but it is not recommended that it be ignored. Once more, Chapter 3, though adding another important viewpoint, is not essential. Chapter 4 is essential, and Chapter 5 is the central chapter on algebraic curves and would be the climax of such a course. A course purely on the theory could stop there, and this material could be covered easily in a semester. However, many students nowadays are interested in the utility of the mathematics they study, and for such students, Chapter 6 is strongly recommended. If time permits, the student can then jump to Section 7.8 to get an introduction to one type of algebraic surface.

Interpolation and Applied Geometry (Introductory)

This would be a course for applied mathematicians, scientists, computer scientists, and engineers. For this course Chapters 1 and 2 are essential, Chapter 3 could be omitted but Chapter 4 should be carefully studied, Chapter 5 should be omitted but the examples and applications in Chapter 6 are very important. The examples in Section 7.5 would give worthwhile exposure to two-dimensional problems and would provide an excellent introduction to the rational surfaces of Section 7.8. Once more, this material could be covered in a semester.

Applied Algebraic Geometry

For the senior student who already has a good mathematical background, a course concentrating on the geometry and applications could be constructed as follows. Most of Chapter 1 could be omitted, but Sections 1.6.1 through 1.6.3 should be studied. Chapter 2 could be glossed over and Chapter 3 omitted. Chapter 4 is also likely to be review. Chapters 5, 6, and 7 would form the main part of the course.

Applied Projective Geometry (a Year Course)

For the serious student wanting to understand the critical points, the entire book is recommended, and it is recommended that it be studied in the sequence presented. It is further recommended that certain things definitely not be omitted.

The finite linear interpolation problem of Chapter 1 is important because it is simple and complete and serves as a striking counterpoint to the nonlinear geometric case. Haar's theorem is the eye-opener.

Chapter 2 takes two steps forward and one back. The geometrical arguments are powerful, but the techniques break down and underline the need for more sophisticated mathematical tools. It is important to appreciate the necessity for more general methods.

Chapter 3 should be stressed. Synthetic methods are often ignored, but it is these very methods that provide insight and are often much more incisive than the algebraic counterparts. Often algebraic solution is impossible without the prior insight gained by the synthetic method.

Chapter 5 gets to the heart of the matter. The theory here is critical. Without it a student cannot fully understand rational curves, and it is these curves that give us the main applications.