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Excerpt
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1
Interpolation

1.1 Introduction

In this chapter we describe what we mean by an interpolation problem, outline the basic theory, show both its generality and its limitations, highlight two-dimensional polynomial interpolation, and find the relationship between this problem and its geometrical counterpart.

The reader is probably familiar with the words "interpolate" and "extrapolate" in the context of finding an approximation to some function value not already tabulated. If the value sought lay between two values already existing in the table then the process of finding the approximation was referred to as "interpolation." Occasionally one sought to extend the table beyond its last entry or add a value before the already existing first value. This process was called "extrapolation." In this book, however, we use the word "interpolation" to refer to a process of passing some curve through given points. When we refer to an *interpolant* we mean a function (or a curve, surface or hypersurface) whose graph passes through a given set of points. We call these points the *interpolation points*. Interpolation is thus distinguished from an *approximation* method, which may not agree with the given function on any set of points but is expected to be "close" to the given function over some interval.

1.2 Polynomial and Rational Interpolation

Let us discuss some examples before we begin to study the theory and applications of the interpolation of data by functions.

1.2.1 Examples

Example 1.1 *(Quadratic polynomial interpolation) Find a quadratic polynomial function that assumes the values* 7 *at* $x = -1$, -1 *at* $x = 1$ and 1 *at* $x = 2$.

We seek a function $p(x) = ax^2 + bx + c$ such that $p(-1) = 7$, $p(1) = -1$, and $p(2) = 1$. Performing the evaluations, we obtain

$$a - b + c = 7,$$
$$a + b + c = -1,$$
$$4a + 2b + c = 1,$$

or, in matrix notation,

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 1 \end{pmatrix}.$$

The determinant of the matrix on the left of the equation has the value -6; hence a unique solution exists (because the determinant is nonzero), namely

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} -1 & 3 & -2 \\ 3 & -3 & 0 \\ -2 & -6 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}.$$

Therefore, the required polynomial is $p(x) = 2x^2 - 4x + 1$. This interpolating parabola is shown in Figure 1.1.

Example 1.2 *(Rational-function interpolation) Find a function of the form*

$$r(x) = \frac{ax + b}{x + c}$$

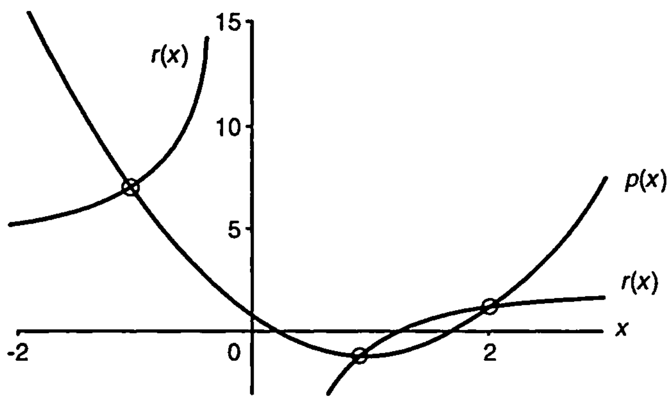

Figure 1.1. Two interpolants through three points.

that assumes the same values at the same points as the polynomial in the previous example, namely $r(-1) = 7$, $r(1) = -1$, *and* $r(2) = 1$.

The interpolation constraints give us

$$\frac{-a+b}{-1+c} = 7, \qquad \frac{a+b}{1+c} = -1, \qquad \frac{2a+b}{2+c} = 1,$$

which in matrix notation (after cross multiplication and rearrangement) is equivalent to

$$\begin{pmatrix} -1 & 1 & -7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 2 \end{pmatrix}.$$

Again the matrix on the left is nonsingular, and the unique solution is given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -2 & -6 & 8 \\ 3 & 15 & -6 \\ -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} -7 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}.$$

Hence the required rational function, shown in Figure 1.1, is

$$r(x) = \frac{3x-4}{x}.$$

1.2.2 Discussion

Both these examples involved a function with three parameters—the a, b, and c in each case—and the corresponding function had to satisfy three conditions. However, in the first example there will be a unique solution regardless of what values of $p(x)$ we were interpolating, whereas in the second example there are infinitely many sets of values for $r(x)$ for which there is no solution. For example, there is no $r(x)$ of the given form that satisfies the conditions $r(-1) = -1$, $r(1) = 3$, and $r(2) = 5$. In this chapter we will concentrate on polynomial interpolation and similar problems, leaving rational interpolation for later chapters.

We note that for Example 1.1 the existence and uniqueness of the solution do not depend on the specific values interpolated, i.e., 7, -1, and 1, since the matrix on the left of the first matrix equation is nonsingular and does not depend on these values. Similarly, as long as the three x-values used for the interpolation are distinct, the matrix concerned is nonsingular and a unique solution exists. This result, stated in more general form, is proved in Example 1.6.

1.3 Definitions

Let us give a few definitions and then proceed with our main discussion of the finite linear interpolation problem.

A *field* F is a set of elements called scalars together with two binary laws of operation on F. These laws are called addition (written $+$) and multiplication (written $\times$). By convention, if no symbol appears between two members of F then the multiplication sign is assumed. Thus, for $\alpha, \beta \in F, \alpha\beta$ is taken to mean $\alpha \times \beta$. For any $\alpha, \beta, \gamma \in F$ the following properties must be satisfied:

1. $\alpha + \beta \in F$. F is closed under addition.
2. $\alpha + \beta = \beta + \alpha$. Addition is commutative.
3. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$. Addition is associative.
4. There exists an element written $0 \in F$ such that $\alpha + 0 = \alpha$.
5. There exists an element called the additive inverse and written $-\alpha$ such that $\alpha + (-\alpha) = 0$. We usually write $\alpha - \alpha = 0$ rather than $\alpha + (-\alpha) = 0$.
6. $\alpha\beta \in F$. Closed under multiplication.
7. $\alpha\beta = \beta\alpha$. Multiplication is commutative.
8. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. Multiplication is associative.
9. There exists an element written $1 \in F$ such that $\alpha 1 = \alpha$.
10. For $\alpha \neq 0$ there exists an element called the multiplicative inverse and written α^{-1} such that $\alpha\alpha^{-1} = 1$.
11. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. Multiplication is distributive with respect to addition.

We will usually work over the field of real numbers R and write $1/\alpha$ or $\frac{1}{\alpha}$ for α^{-1}.

A *vector space* V over a field F is a set of elements called vectors together with two binary laws of operation. One of these laws is called vector addition (or simply, addition) and operates on two vectors. The second binary law, called scalar multiplication, operates on a scalar and a vector. The vector addition is written $+$. If $\alpha \in F$ and $f \in V$ then αf is taken to mean scalar multiplication between the scalar α and the vector f. We do not define a multiplication between two vectors. For any $f, g, h \in V$ and $\alpha, \beta \in F$, the following properties must be satisfied:

1. $f + g \in V$. V is closed under addition.
2. $f + g = g + f$. Addition is commutative.
3. $f + (g + h) = (f + g) + h$. Addition is associative.
4. There exists an element written 0 and belonging to V such that $f + 0 = f$.
5. For each $f \in V$ there exists an element of V called the additive inverse and

written $-f$ such that $f + (-f) = 0$. We usually write $f - f = 0$ rather than $f + (-f) = 0$.

6. $\alpha f \in V$. V is closed under scalar multiplication.
7. $1f = f$.
8. $\alpha(f + g) = \alpha f + \alpha g$. Scalar multiplication is distributive with respect to vector addition.
9. $(\alpha + \beta)f = \alpha f + \beta f$. Scalar multiplication is distributive with respect to addition in the field.

When the field is the field of real numbers, we will refer to the vector space as a *real vector space*.

A subset S of a vector space V is said to be a *subspace* if it is also a vector space over the same field as V. A subset S will be a subspace if it is closed under addition and scalar multiplication and contains the zero vector (Exercise 1.12). For example, the set $S = \{(x, x) : x \in R\}$ is a subspace of the two-dimensional real vector space $R_2 = \{(x, y) : x, y \in R\}$ over the field R, with the usual laws for scalar multiplication and vector addition.

Let $f_i \in V, i = 1, 2, \ldots, n$.

The set $\{f_i : i = 1, 2, \ldots, n\}$ is said to *span* V if, for each $f \in V$, there exists a set of scalars $\alpha_i \in F$ such that

$$f = \sum_{i=1}^{n} \alpha_i f_i.$$

The set $\{f_i : i = 1, 2, \ldots, n\}$, is said to form a *linearly independent* set if the equation

$$\sum_{i=1}^{n} \beta_i f_i = 0, \qquad \beta_i \in F, \quad i = 1, 2, \ldots, n,$$

implies $\beta_i = 0, i = 1, 2, \ldots, n$.

If a set of vectors is not linearly independent, then it is said to be *linearly dependent*. We often omit the qualifier "linearly" and refer to a set of vectors as being dependent or independent.

If a set of linearly independent vectors $\{f_i : i = 1, 2, \ldots, n\}$ spans V, then the set forms a *basis* for V and the *dimension* of V is n. In this situation each coefficient $\alpha_i, i = 1, 2, \ldots, n$, referred to above is unique.

When the vector space is a space of real-valued functions over some subset S of reals, then the statement

$$\sum_{i=1}^{n} \beta_i f_i = 0$$

may be written $\sum_{i=1}^{n} \beta_i f_i(x) = 0$, and is taken to mean

$$\sum_{i=1}^{n} \beta_i f_i(x) = 0 \qquad \text{for all} \quad x \in S \subseteq R,$$

where S is the particular subset of the reals that is under consideration.

Example 1.3 *Examine the dependence of the vectors $f_1 = (1\ \ 0\ \ 1\ \ 0)^T$, $f_2 = (0\ \ 1\ \ 0\ \ 1)^T$, $f_3 = (0\ \ 1\ \ 1\ \ 0)^T$, and $f_4 = (1\ \ 0\ \ 0\ \ 1)^T$ in R_4, where the superscript T denotes the transpose of the (row) vector.*

We set

$$\beta_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \beta_2 \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \beta_3 \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \beta_4 \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

that is,

$$\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\\1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{pmatrix} \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Proceeding by Gaussian elimination, we get

$$\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\\0 & 0 & 1 & -1\\0 & 0 & 0 & 0\end{pmatrix} \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

We can choose β_4 arbitrarily; then $\beta_3 = \beta_4$, $\beta_2 = -\beta_4$, and $\beta_1 = -\beta_4$. Hence $\sum_{i=1}^{n} \beta_i f_i(x) = 0$ does not imply that $\beta_i = 0$ for $i = 1, 2, 3, 4$, and the given vectors are dependent.

Example 1.4 *Let $V = P_3(x)$, the space of polynomials of degree three in a single variable x. Let $S_1 = \{0, \frac{1}{2}, 1\}$ and $S_2 = \{x : 0 \le x \le 1\}$. Show that $f_1 = 1$, $f_2 = x$, $f_3 = x^2$, and $f_4 = x^3$ are dependent on S_1 but independent on S_2.*

Since in this example $\sum_{i=1}^{4} \beta_i f_i$ is a polynomial of degree three, we seek a polynomial of degree three that is zero on S_1. The polynomial $p(x) = \alpha x(x - 1)(2x - 1)$ satisfies this condition. Hence $\sum_{i=1}^{4} \beta_i f_i = 0$ if $\beta_1 = 0$, $\beta_2 = \alpha$,

$\beta_3 = -3\alpha$, and $\beta_4 = 2\alpha$. Therefore, the functions $f_i, i = 1, \ldots, 4$, are dependent on S_1.

To show independence on S_2, select any four (or more) distinct points, for example $0, \frac{1}{2}, 1$, and x_4, where $x_4 \in S_2$ but $x_4 \neq 0, \frac{1}{2}$, or 1. A cubic which is zero at $x = 0, \frac{1}{2}, 1$ must be of the form $p(x) = \alpha x(x-1)(2x-1)$. Then $p(x_4) = 0$ implies that $\alpha x_4(x_4 - 1)(2x_4 - 1) = 0$, so that $\alpha = 0$. Therefore the f_i are independent on S_2.

Example 1.5 *Show that x and* $\tan x$ *are linearly independent solutions of the differential equation*

$$y''(\tan x - x\sec^2 x) + 2(y'x - y)\tan x \sec^2 x = 0, \qquad x \in (0, \pi/2).$$

Let $y_1 = x$; then $y_1' = 1$ and $y_1'' = 0$. Substitution shows that $y_1 = x$ satisfies the differential equation and hence x is a solution.

Let $y_2 = \tan x$; then $y_2' = \sec^2 x$ and $y_2'' = 2\sec^2 x \tan x$. Substitution shows that $y_2 = \tan x$ satisfies the differential equation.

Suppose $ax + b\tan x = 0$ for all $x \in (0, \pi/2)$. Then choosing $x = \pi/4$ and $x = \pi/3$, we get

$$\begin{pmatrix} \pi/4 & 1 \\ \pi/3 & \sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies $a = b = 0$. Hence x and $\tan x$ are linearly independent on $x \in (0, \pi/2)$.

We introduce a few more definitions.

A *functional* L over a vector space V is a mapping from V to the field F that associates with each vector f of V a unique scalar in F designated by $L(f)$. The functional is said to be a *linear functional* if, in addition, we have

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g) \qquad \text{for} \quad f, g \in V \;\text{ and }\; \alpha, \beta \in F.$$

For example, consider the two-dimensional vector space R_2 over the field of reals. We define a functional $L((x, y)) \equiv x + y \in R$. Then

$$\begin{aligned} L(\alpha(x_1, y_1) + \beta(x_2, y_2)) &= L((\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)) \\ &= L((\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)) \\ &= (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) \\ &= \alpha(x_1 + y_1) + \beta(x_2 + y_2) \\ &= \alpha L((x_1, y_1)) + \beta L((x_2, y_2)), \end{aligned}$$

so that our functional is linear.

1.3.1 The Dual Space

For two linear functionals L_1 and L_2 and a scalar α, we can define an addition between the linear functionals and a scalar multiplication between a scalar and a linear functional in the following way:

$$(L_1 + L_2)(f) = L_1(f) + L_2(f), \qquad f \in V,$$
$$(\alpha L_1)(f) = \alpha L_1(f), \qquad f \in V, \quad \alpha \in F.$$

With such definitions of addition and scalar multiplication the set of linear functionals over V forms a vector space called the *dual space* (or conjugate space) and denoted by V^*. The zero linear functional is the functional that maps each vector of V onto the zero of the field. We have the following result.

Theorem 1.1 *If V has dimension n then so does V^*.*

Proof. Let $\{v_i : i = 1, 2, \ldots, n\}$ be a basis for V. Then

$$f \in V \quad \Rightarrow \quad f = \sum_{i=1}^{n} \alpha_i v_i,$$

where, given f, the α_i are unique scalars. Define $\{L_i : i = 1, 2, \ldots, n\}$ by

$$L_i(f) = \alpha_i.$$

Then the $L_i i = 1, \ldots, n$, are linear functionals (Exercise 1.12). To show that they are linearly independent, suppose that $\sum_{i=1}^{n} \beta_i L_i = 0$. Then

$$\sum_{i=1}^{n} \beta_i L_i(f) = 0 \qquad \text{for all} \quad f \in V.$$

In particular,

$$\sum_{i=1}^{n} \beta_i L_i(v_j) = 0, \qquad j = 1, 2, \ldots, n.$$

By definition of the L_i we have $L_i(v_j) = \delta_{ij}$. Therefore $\beta_j = 0$, $j = 1, 2, \ldots, n$, and hence the L_i are linearly independent. The dimension of V^* is therefore at least n.

Let us suppose that V^* has dimension greater than n. Then there must exist a set of $n + 1$ linearly independent linear functionals $L_1, L_2, \ldots, L_{n+1}$. The set of equations

$$\sum_{i=1}^{n+1} \beta_i L_i(v_j) = 0, \qquad j = 1, 2, \ldots, n,$$

is underdetermined and thus possesses a nontrivial solution. For these solution values β_i (not all zero) we have

$$\begin{aligned}\sum_{i=1}^{n+1} \beta_i L_i(f) &= \sum_{i=1}^{n+1} \beta_i L_i\left(\sum_{j=1}^{n} \alpha_j v_j\right) \\ &= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n+1} \beta_i L_i(v_j)\right) \\ &= 0.\end{aligned}$$

Therefore, $\sum_{i=1}^{n+1} \beta_i L_i$ is the zero linear functional, and hence the set $\{L_i : i = 1, 2, \ldots, n+1\}$ is linearly dependent. This contradicts our assumption that the dimension of V^* is greater than n. Therefore, the dimension of V^* is n. ■

1.4 The Finite Linear Interpolation Problem

We are now in a position to define the simplest kind of interpolation problem, namely the *finite linear interpolation problem*. This problem is stated as follows: Given a vector space V of finite dimension n, a set of linear functionals $L_i, i = 1, 2, \ldots, n$, in the dual space V^* of V, and a set of scalars $\alpha_i, i = 1, 2, \ldots, n$, in the field F, find an $f \in V$ such that $L_i(f) = \alpha_i$ for $i = 1, 2, \ldots, n$.

Let us return to Example 1.1. Polynomials of degree two with the usual addition and multiplication form a real vector space of dimension three, which we denote by $P_2(x)$. Let us define

$$L_1(f) = f(-1), \qquad L_2(f) = f(1), \qquad L_3(f) = f(2),$$

$$\alpha_1 = 7, \quad \alpha_2 = -1, \quad \text{and} \quad \alpha_3 = 1.$$

We can now restate the problem in the terminology of the finite linear interpolation problem, as folllows: find a function $f(x) \in P_2(x)$ such that $L_i(f) = \alpha_i$ for $i = 1, 2, 3$.

We are interested both in determining under what conditions there exists a unique solution to the finite linear interpolation problem and in finding that solution. The solution, when it exists, can always be written as a linear combination of basis vectors. The first step is to determine whether or not the given linear functionals form a basis for V^*.

1.4.1 Matrix Nonsingularity

Theorem 1.2 *Let V be a vector space of dimension n, $\{v_i : i = 1, 2, \ldots, n\}$ a set of vectors in V, and $\{L_i : i = 1, 2, \ldots, n\}$ a set of linear functionals in V^*.*

If the sets $\{v_i\}$ *and* $\{L_i\}$ *are independent in* V *and* V^**, respectively, then the determinant*

$$\det(L_i(v_j)) \neq 0.$$

Conversely, if $\det(L_i(v_j)) \neq 0$ *and one of the sets* $\{v_i\}$ *or* $\{L_i\}$ *is independent, then the other set is also independent.*

Proof. We will assume that $\det(L_i(v_j)) = 0$ and show that the $\{L_i\}$ must be dependent, giving us a contradiction. We want, therefore, to show that there exists a set of $\{\alpha_i : i = 1, 2, \ldots, n\}$, not all elements zero, such that $\sum_{i=1}^{n} \alpha_i L_i = 0$. The matrix $(L_i(v_j))$ is singular; therefore there exists a nontrivial solution to the n equations

$$(\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n)(L_i(v_j)) = (0 \quad 0 \quad \cdots \quad 0). \tag{1.1}$$

That is, we have a set of $\{\alpha_i\}$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i L_i(v_j) = 0, \qquad j = 1, 2, \ldots, n.$$

Therefore, since the v_i, $i = 1, 2, \ldots, n$, are independent, $\sum_{i=1}^{n} \alpha_i L_i$ must be the zero linear functional, and hence the $\{L_i\}$ must be dependent, providing us with our contradiction. Hence $\det(L_i(v_j)) \neq 0$.

Conversely, assume that $\det(L_i(v_j)) \neq 0$. Then the only solution to the system of equations (1.1) is the zero solution. That is, the only linear functional of the form $\sum_{i=1}^{n} \alpha_i L_i$ such that $\sum_{i=1}^{n} \alpha_i L_i(v_j) = 0$, $j = 1, 2, \ldots, n$, is the one given by $\alpha_i = 0$, $i = 1, 2, \ldots, n$. If the set $\{v_i\}$ is independent then this functional will map all vectors in V to zero and hence it is the zero linear functional. Therefore,

$$\sum_{i=1}^{n} \alpha_i L_i = 0 \quad \text{implies} \quad \alpha_i = 0, \quad i = 1, 2, \ldots, n,$$

and hence the $\{L_i\}$ are independent.

If the given $\{L_i\}$ are independent and $\det(L_i(v_j)) \neq 0$, then the only solution of

$$(L_i(v_j)) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is $\beta_i = 0$, $i = 1, 2, \ldots, n$. In this case, since the only vector in V to be mapped onto zero by every member of a basis for V^* must be the zero vector, we